Expansive Dynamical Systems

Doctoral Thesis

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Abstract

It is a thesis about dynamical systems with some kind of expansiveness. We consider homeomorphisms and flows on compact metric spaces. The smooth category is considered and some results are proved for manifolds. Several variations of expansiveness are considered. In the discrete time case we consider: cw-expansiveness, N-expansiveness, hyper-expansiveness. For the case of continuous flows we study: geometric and kinematic expansiveness, positive expansiveness and robust expansiveness. The results we obtained were or will be published in [6–10].

Resumen

Esta tesis versa sobre sistemas dinámicos con diversos tipos de expansividad. Consideramos homeomorfismos y flujos en espacios métricos compactos. También se considera la categoría diferenciable y algunos resultados se demuestran en variedades. Diferentes variantes de la expansividad son tomados en cuenta. En tiempo discreto: cw-expansividad, N-expansividad, hiperexpansividad. En el caso de flujos: expansividad cinemática y geométrica, expansividad positiva y expansividad robusta. De los resultados obtenidos algunos fueron y otros serán publicados en las referencias [6–10].

Contents

Chapter 1

Introduction

In this work we will consider several forms of expansivity and different results are proved. Let us remark some of the results that we have obtained in this Thesis. We characterize hyper-expansive homeomorphisms. We prove that 2-expansive homeomorphisms on surfaces without wandering points are expansive. We prove the non-existence of smooth kinematic expansive suspensions of irrational rotations. We prove that positive expansive flows are supported on a finite number of perodic orbits.

Let us describe the contents of the Thesis. We start in the next chapter with a panoramic view of the theory of expansive systems. We review, from our viewpoint, the main results from 1950 until present days. This chapter is the result of several years of bibliographical research. The purpose is to explain why the idea of expansivity started, how it was developed, which are the guiding questions and where is it going now.

In Chapter 3 the meaning of expansiveness is investigated by showing several classical and well known equivalent definitions. One of this equivalences is due to Lewowicz and is related with Lyapunov functions. We present a new approach to this topic by introducing Whitney's size functions for the construction of Lyapunov functions for isolated sets. Since expansivity and cw-expansivity are related with isolated sets, applications to these systems are given. Other known variations of expansivity are presented and examples are given. The whole chapter is seen from the viewpoint of isolated sets, it is my opinion that this simplifies the exposition.

In Chapter 4 we first review known results of expansive and cw-expansive homeomorphisms. These results are related with Lyapunov stable points, stable sets and topological dimension. In the second section we study hyper-expansiveness, that is, the expansiveness of the induced homeomorphism in the space of compact subsets equipped with the Hausdorff metric. We give a characterization of such systems. In the final section we introduce a new variation of expansiveness that we call (m, n)-expansiveness. Several basic properties are obtained.

In Chapter 5 we apply Lewowicz's techniques in the study of expansive surface homeomorphisms. We prove that expansiveness is equivalent to cw-expansiveness in the absence of bi-asymptotic sectors. In the second section we prove that 2-expansiveness implies expansiveness if the non-wandering set is the whole surface. An example is given in the third section to prove that there are 2-expansive surface homeomorphisms that are not expansive.

In Chapter 6 we consider expansive flows from a kinematic and a geometric viewpoint. Several technical variations are considered with respect to the kind of time-reparametrizations allowed. The hierarchy of this definitions is studied on compact metric spaces and on compact surfaces. Kinematic expansiveness of suspensions and surface flows is studied.

In Chapter 7 we consider positive expansive flows. In the kinematic framework we study basic properties of such flows especially on surfaces. For positive geometric expansive flows we prove that they consist of a finite number of compact orbits.

In Chapter 8 perturbations of kinematic expansive flows are considered in the C^1 category. In the first section we consider conservative vector fields in the annulus. In the second one we prove that robust kinematic expansiveness is equivalent to robust geometric expansiveness in the absence of singularities.

Chapter 2

A panoramic view

In this chapter we will review the development of the theory of expansive systems.

2.1 Unstable dynamics

In this section we present the first results, the main questions and the main variations of the definition of expansive system.

2.1.1 Unstable homeomorphisms.

The study of expansive homeomorphisms started in 1950 when W. R. Utz [128] defined these systems with the name of *unstable homeomorphisms*. In this first article some general properties were proved related with asymptotic trajectories, the cardinality of the set of periodic points and the powers of expansive homeomorphisms.

In Utz's paper the examples on compact spaces were subdynamics of shift maps defined on Cantor sets. It seems to be the case that Utz's motivation for the definition of expansive homeomorphism was to generalize this kind of systems. In the references of the paper one finds a book on general topology by W. Sierpinski and four works on symbolic dynamics by the authors M. Garcia, W. H. Gottschalk, G. A. Hedlund and M. Morse that dates from 1938 to 1948. See [37, 38, 48, 49]. These papers can be considered as part of the foundation of abstract symbolic dynamics.

In order to illustrate the kind of problems studied in topological dynamics before Utz's paper, let us recall a question raised by Birkhoff [15] in 1936: given a minimal set and two points in the set, does there exist an orbit preserving homeomorphism of the minimal set onto itself transforming one of these points into the other? In [49] Hedlund gave a negative answer. He considered a Sturmian minimal subshift with two asymptotic points. Therefore we have that Utz's result on the existence of asymptotic orbits of expansive homeomorphisms is a generalization of Hedlund's result.

We can say that the theory of expansive homeomorphisms started based on symbolic dynamics but it quickly developed by itself. One of the main questions of the theory was present since the beginning:

What compact metric spaces can carry an expansive homeomorphism?

In 1954 B. Bryant [21] proved that the interval does not admit such systems and raised another natural question:

Are there expansive homeomorphisms on connected spaces?

In his note of 1955, R. F. Williams [137] proved that the *two-solenoid* is expansive. This was the first continuum shown to admit an expansive dynamic. In 1960 J. F. Jakobsen and W. R. Utz [57] proved that the compact 2-dimensional disc does not admit expansive homeomorphisms; in fact they showed that the circle does not admit such dynamics. In this way it was proved that no compact one-dimensional manifold can carry an expansive homeomorphism. In 1962 B. Bryant [22] discovered the property of uniform expansiveness that would be very useful in future works. Other general properties were proved in this and other articles and a new question gained in interest:

May a locally connected space admit an expansive homeomorphism?

The only known examples at this time were the shift map and the two-solenoid. These examples are supported on non-locally connected spaces. No manifold was known to admit an expansive homeomorphism.

The theory will grow with more examples, specially from the hyperbolic dynamics. But it will be also expanded from another viewpoint, several variations of the definition will follow the development through the years. Let us in the next section review this topic.

2.1.2 Some variations of expansiveness.

The definition of expansiveness has shown to have a *robust interest*. That is, small variations on the definition are also interesting. The first of such variations appeared in 1952 when S. Schwartzman [118] considered what now is called *positive expansiveness*. His definition requires that different points are separated in positive time. He proved that the only compact metric spaces admitting such homeomorphisms are finite sets.

In 1970 W. L. Reddy [107] introduced *point-wise expansiveness*, a variation that does not require the existence of a uniform expansive constant but each point has a positive one. He proved that even on a compact space, point-wise expansiveness does not imply expansiveness. In spite of this he generalized results for this weaker definition.

Another generalization of expansiveness was introduced in 1972 [18] by R. Bowen called *entropy-expansiveness* or *h-expansiveness*. The definition requires that if a set has small diameter for all the time then it has vanishing entropy. Some results concerning the entropy of an expansive homeomorphism were developed in this paper.

W. Bauer and K. Sigmund in 1975 [13] considered the relationship between a homeomorphism on a compact metric space and its induced action on the space of probability measures and the space of compact subsets called *hyperspace*. They proved that the induced homeomorphism on the probabilities is expansive if and only if the original space is finite. For the action on the hyperspace they gave some examples proving that the expansiveness of the homeomorphism does not imply the expansiveness of the induced homeomorphism.

In 1993 [61] H. Kato defined *continuum-wise expansiveness* by requiring that if a continuum has small diameter for all the time then it is a singleton. This definition seems to be based on the techniques developed for expansive homeomorphisms, it was designed in order to be able of extending important results.

In 2011 C. A. Morales [85] considered *measure-expansiveness* by requiring that the probability of two orbits remain close each other for all time is zero. He extended results of expansive systems on compact metric spaces to the measure-expansive context. In 2012 the same author [87] defined another variation called *N-expansiveness*. Now a set of points whose orbits are close for all the time has cardinality smaller than N. There, some results of expansive homeomorphisms are generalized to this context.

2.2 Hyperbolic systems

2.2.1 Fundamental examples.

The existence of expansive homeomorphisms on continua was first proved by W. L. Reddy [106]. In 1965 he showed that the torus of dimension greater than one admits expansive homeomorphisms.

In 1967 a fundamental paper in dynamical systems of D. V. Anosov [1] appeared. A generalization of the geodesic flow of a compact manifold of negative curvature was developed in this work. This generalization is now called as *Anosov systems*. They are characterized by a uniform hyperbolicity of the tangent map on the whole ambient manifold. He used the property of expansiveness to show that such systems are structurally stable.

At the same year another fundamental paper in dynamical systems was published. In [121] S. Smale developed the theory of hyperbolicity of invariant sets of diffeomorphisms and introduced the definition of *Axiom A* diffeomorphisms. Expansiveness has shown to be a very important property of such sets.

In 1970 T. O'Brien and W. L. Reddy [94] showed that the surfaces of positive genus admit expansive homeomorphisms. These examples are now known as pseudo-Anosov diffeomorphisms. With all these examples (Anosov systems, hyperbolic sets and pseudo-Anosov diffeomorphisms) the theory of expansive systems was highly enriched. Of course the theory of dynamical systems too.

2.2.2 Expansive flows.

A fundamental work for the theory of expansive flows, written by R. Bowen and P. Walters, appeared in 1972 [17]. In this paper they made a careful analysis of the definition in the context of continuous flows on compact metric spaces, and gave extensions of results known for Anosov systems, mainly related with the topological entropy. As we said, the expansiveness of flows was first considered by Anosov, and therefore it is natural that such flows have no singular (i.e. equilibrium) points because the motivation of Anosov was the study of geodesic flows that usually are restricted to the unit tangent bundle. Bowen and Walters found that in order to extend known results from expansive homeomorphisms to expansive flows, the definition has to involve the use of time reparametrizations of single trajectories. In 1979 H. B. Keynes and M. Sears [66] extended the definition of expansive flows considering different families of time reparametrizations.

In 1984 A. A. Gura [40] discovered another kind of expansiveness of flows. He proved that the horocycle flow of a surface of negative curvature is separating[1] in both directions of time and in a strong sense. Separating means that points on different global orbits are separated by the the flow to a fixed separating constant. He proved that this separation occurs in positive and in negative times. Moreover, he proved that these properties are shared with every global time change of the flow. It is known that the horocycle flow is not expansive in the sense of Bowen and Walters. In 1998 [29] A. DeStefano and G. Hall presented a separating flow on the two-dimensional torus. It is a time change of a minimal flow. Another example was recently given S. Matsumoto in [81].

As we said, the definition of expansive flow of Bowen and Walters does not admit singular points. In 1984 M. Komuro [69], interested in the Lorenz attractor, introduced a different definition called k^*-*expansiveness*. It is known that the Lorenz attractor has a singular point accumulated by regular orbits. Therefore it is not expansive in the sense of Bowen and Walters, but it is k^*-expansive as proved by Komuro.

[1] It is interesting to note that Gura used the term *separating* for *expansive*. In the case of flows, the separation property proved by Gura in the horocycle flow is different from Bowen-Walters definition.

2.2.3 Robust expansiveness.

In 1975 R. Mañé characterized the diffeomorphisms that are robustly expansive in the C^1 topology as what he called *quasi-Anosov diffeomorphisms*. At this time Mañé asked if every quasi-Anosov diffeomorphism is in fact an Anosov one. The answer appeared one year later, in the context of vector fields, when C. Robinson [110] found a quasi-Anosov flow that is not Anosov on a eleven dimensional manifold. This example was simplified in the same year by J. Franks and C. Robinson [32]. They constructed a quasi-Anosov diffeomorphism on a three-dimensional manifold that is not Anosov. This was the first example of an expansive homeomorphism on a three-dimensional manifold with wandering points.

Mañé's result for robustly expansive diffeomorphisms were generalized for continuum-wise expansiveness by K. Sakai in 1997 [117]. The case of (Bowen-Walters) expansive C^1 vector fields was considered by K. Moriyasu, K. Sakai and W. Sun in 2005 [88].

2.2.4 Topological dimension.

Mañé was also interested in expansive systems from a topological viewpoint. In 1979 [78] he proved that if a compact metric space admits an expansive homeomorphism then its topological dimension is finite. In 1989 A. Fathi gave another proof [30] with different techniques. In [78] Mañé also showed that the only spaces admitting minimal expansive homeomorphisms are totally disconnected, or equivalently has vanishing topological dimension. This result extends the corresponding one in the setting of hyperbolic diffeomorphisms and flows previously proved by R. Bowen [18,19]. Mañé's proof was by contradiction, and assuming the existence of a non-trivial continuum he was able to construct non-trivial connected stable sets. His techniques were very important in the development of the theory.

In 1981 H. B. Keynes and M. Sears [67] extended these results for flows using the definition of Bowen and Walters. They proved that if a compact metric space admits an expansive flow then its topological dimension is finite. If in addition it is a minimal flow and has no spiral orbits then the topological dimension of the space is at most one (i.e. local cross sections have dimension zero). In the case of homeomorphisms a spiral point gives rise to a periodic point but in the case of flows, since the definition considers reparametrizations, the conclusion is not clear. It is still an open problem of the theory.

In 1993 H. Kato [61] extended Mañé's proofs for cw-expansive homeomorphisms.

2.2.5 Lyapunov functions and stable points

In 1892, Lyapunov [76] studied the problem of stability of solutions of differential equations. He developed a theory of stability and extended the notion of *energy function* to what now are called Lyapunov functions. He proved that the existence of a strictly decreasing Lypunov function for a equilibrium point implies its asymptotic stability. In 1949 J. L. Massera [79] proved the converse result by constructing a Lyapunov function for such points.

In 1906 [33] M. Frechet published a fundamental work in topology. For example, the concept of metric spaces were introduced there. But we wish to mention that he considered a special metric in the space of curves. To measure the distance between two curves he considered the infimum in all the reparametrizations of the sup-distance of the curves. In 1964 Massera [80] considered the problem of stability of trajectories. He considered several variations in the definition, being one of them stated using the Frechet-distance of curves.

In 1978 [26] C. Conley developed the theory of global Lyapunov functions and isolated sets. Some authors referred to his results as the *Fundamental theorem of dynamical systems*. The relationship with expansiveness is that expansiveness is equivalent with the diagonal being isolated for the product homeomorphism. Therefore, the construction of Lyapunov functions for isolated sets can be applied to expansive homeomorphisms.

In 1980 J. Lewowicz [72] introduced the techniques of Lyapunov functions for the study of topological stability and expansive systems. He proved that a diffeomorphism is Anosov if and only if there is a non-degenerate Lyapunov quadratic function in the tangent bundle. He also considered Lyapunov functions for the problem of topological stability. Following Massera's techniques he proved that expansiveness is equivalent with the existence of a Lyapunov function. Such functions will be a fundamental tool in his future works on expansive homeomorphisms.

In 1990 R. Ures [127] gave another construction of a Lyapunov function for an expansive homeomorphism. In 1993 M. Paternain [100] extended this constructions for expansive flows on manifolds. He also proved that expansive flows on manifolds have no stable points in the sense of Frechet-Massera.

2.2.6 Hyperbolic metrics.

In 1989 A. Fathi [30] was able to construct a special metric for an expansive homeomorphism on a compact metric space. It has a hyperbolic behavior and extends previous constructions by W. L. Reddy [109]. In [30] the hyperbolic metric is used to: 1) give a new proof of Mañé's result, proving that if a compact metric space admits an expansive homeomorphism then its topological dimension is finite and 2) prove that every expansive homeomorphism defined on a compact metric space with positive topological dimension has positive topological entropy. This result was extended by Kato [61] (of course, without using hyperbolic metrics) for cw-expansive homeomorphisms. Hyperbolic metrics can also be used to construct Lyapunov functions.

2.3 Expansiveness on manifolds

2.3.1 Continua and hyperspace.

For hyperbolic diffeomorphisms the stable manifold theorem is a very powerful result. In this setting one starts with a model for local stable sets: the linear stable subspaces for the tangent map. One concludes that local stable sets are embedded manifolds. In the case of expansive homeomorphisms on manifolds one of the main problems is to determine the topological structure of stable sets. Some results, specially in low dimensions were developed. The techniques used for this purpose are related with continua theory, a very interesting branch of general topology. Specially important are the results characterizing Euclidean spaces.

Recall that a continuum is a compact connected metric space. According to Charatonik [25][2], the definition is due to G. Cantor [23]. Hyperspace theory has its beginning with the work of F. Hausdorff and L. Vietoris. Given a topological space, the hyperspace is the space of all its closed subsets equipped with the Vietoris topology. For compact metric spaces, the Vietoris topology can be defined with the Hausdorff metric introduced in 1914 in his fundamental book [45]. This metric is very important in the study of expansive homeomorphisms, for example stable continua on Peano spaces are constructed taking limits in this distance.

The problem of disconnecting the plane by continua was studied by Z. Janiszewski. In 1913 [58] he proved that if the intersection of two planar continua neither of which disconnects the plane is connected, then their union also does not disconnect the plane. Janiszewski's result is applied in the study of expansive surface homeomorphisms. Essentially it allow us to think of stable continua as if they were curves (a fact that is later proved).

Around 1913 it has been shown by S. Mazurkiewicz [82,83] and H. Hahn [42,43] that a metric continuum is locally connected if and only if it is a continuous image of the unit closed interval. This result reduces the problem of proving arc-connection to prove the local connection, a key step in the study of expansive surface homeomorphisms.

In 1931 Mazurkiewicz [84] proved that the hyperspace of a space with positive dimension has infinite dimension. This result combined with Mañé's result on the dimension of a space admitting an expansive homeomorphism gives us that the expansiveness in the hyperspace of the induced homeomorphism implies that the original space has dimension zero.

In 1933 H. Whitney [135] made two contributions that we wish to remark. The first one is of a topological nature. He introduced what now are called *size functions*. They are continuous functions defined on the hyperspace that measure the *size* of a compact set. The main feature is that these functions are increasing with respect to inclusions of sets. Applications in continuum theory were found later as can be seen in Nadler's book [92]. His motivation was to parameterize

[2] The interested reader should consult [25], as we did, for more on the history of continuum theory.

a regular family of curves to obtain a flow. The second result that we wish to remark from Whitney's paper is the construction of local cross sections for continuous flows on metric spaces. He gave a very simple construction, and later it was a very useful technique in the study of expansive flows.

2.3.2 Plane continua

A plane continua is a compact connected subset of the Euclidean plane. Those spaces can be classified according to its topological dimension and the number of components of its complement. A natural problem is to determine which plane continua admit expansive homeomorphisms. The first results in this direction were proved in 1954 [21] and 1960 [57] when Bryant, Jacobsen and Utz proved that the interval and the circle do not admit expansive homeomorphisms.

The first example, to our best knowledge, of a plane continuum admitting an expansive homeomorphism is the attractor introduced by Plykin in 1984 [105]. It is a one-dimensional plane continua with four components in its complement. In 1990 Kato [59] proved that plane Peano continua do not admit expansive homeomorphisms, generalizing the results for the interval and the circle.

The result for the interval was generalized in another direction by Mouron in 2002 [89] by proving that if a one-dimensional plane continuum does not separate the plane, then it does not admit expansive homeomorphisms. In 2003 [90] he constructed a two-dimensional plane continuum admitting expansive homeomorphisms. The same author in 2008 [91] extended the result for the circle by showing that if a one-dimensional plane continuum separates the plane in two components, then it does not admit an expansive homeomorphism.

2.3.3 Expansiveness on surfaces.

One of the first problems of the theory was to determine if the spheres admit expansive homeomorphisms. As we said, the one-dimensional case was solved in 1960 [57]. The two-dimensional case was harder to solve. The first result to our best knowledge, is by P. Lam [71]. In 1970 there it is proved that there is no orientation preserving expansive homeomorphism on the 2-sphere with exactly one fixed point.

In 1988 K. Hiraide [50] proved that every expansive surface homeomorphism with the pseudo-orbit tracing property is conjugate to a hyperbolic toral automorphism (a linear Anosov diffeomorphism).

In 1989 J. Lewowicz [74] and in 1990 K. Hiraide [52] proved that the 2-sphere does not admit expansive homeomorphisms and moreover, they showed that expansive homeomorphisms of surfaces are pseudo-Anosov. Their proofs were based on a very nice study of the topology of stable sets. They were developed independently but can be divided in two parts being the first one with some similarities. They first construct stable and unstable singular foliations. For this they developed a topological stable manifold theorem, special for surfaces. The key point in both works is to prove that local stable sets are locally connected in order to conclude the arc-connection.

Then, in Lewowicz's paper it is proved that the two-dimensional sphere does not admit expansive homeomorphisms with an argument of the Poincaré-Bendixon theory of surface flows. In Hiraide's article it is applied an index argument. The case of surfaces of higher genus, in Lewowicz's work is considered via universal coverings and proving that the expansive homeomorphism is conjugated to a pseudo-Anosov diffeomorphism. In Hiraide's paper, it is directly constructed two invariant measures, expanding and contracting, with arguments from the interval exchange maps theory.

In 1991 L. F. He and G. Z. Shan [47] showed that no surface admits expansive flow without singular points. Singular expansive flows of surfaces were later studied in [5].

2.3.4 Expansive homeomorphisms on three-manifolds.

In 1989 K. Hiraide [51] proved that expansive homeomorphisms of n-tori with the pseudo-orbit tracing property are conjugate to hyperbolic toral automorphisms.

In 1993 J. L. Vieitez [130] proved that an expansive homeomorphism of a compact three-dimensional manifold with a dense set of topologically hyperbolic periodic points has a local product structure defined on an open invariant dense subset of the manifold. In 1996 [132] Vieitez showed, under the same hypothesis, that the manifold is a torus and the homeomorphism is conjugate to a linear Anosov isomorphism. Generalizations of these results were given later in [4].

In 1996 [131] J. L. Vieitez considered expansive diffeomorphisms on three-manifolds without wandering points. Assuming also that there is a hyperbolic periodic point with a homoclinic intersection he proved that the diffeomorphism is conjugate to an Anosov isomorphism of the torus.

In 2002 J. L. Vieitez [133] proved that on three-dimensional manifolds there are no *pseudo-Anosov* diffeomorphisms by showing that the only expansive $C^{1+\theta}$-diffeomorphisms on three manifolds without wandering points are Anosov diffeomorphisms on the torus.

2.3.5 Expansive flows on three-manifolds.

In 1990 T. Inaba and S. Matsumoto [56] and also M. Paternain [100] considered expansive flows on three-manifolds and extended the results of Lewowicz and Hiraide of expansive surface homeomorphisms. They proved the existence of a stable and an unstable foliation with a finite number of singular periodic orbits. In [100] Paternain also generalized a previously known result for Anosov flows. He proved that if a three-manifold admits an expansive flow then its fundamental group has exponential growth. In particular, the three dimensional sphere does not admit expansive flows without singular points. In this paper it is also proved that expansive flows on manifolds has no stable points and Lyapunov functions are constructed.

In 1993 M. Brunella [20] proved that expansive flows on a three-manifold which is a Seifert fibration of a torus bundle over the circle are topologically equivalent to a transitive Anosov flow.

2.4 Geodesic flows and homoclinic classes

The study of homoclinic orbits and geodesic flows can be considered as the foundation of chaotic dynamical systems.

2.4.1 Geodesic flows

Geodesic flows of surfaces with negative curvature were Hadamard's motivation for introducing symbolic dynamics and these abstract systems were the examples that Utz generalized when he started the study of expansive homeomorphisms. Also, geodesic flows were Anosov's motivation for studying globally hyperbolic diffeomorphisms. Therefore, it is natural that the theory of expansive systems turns its focus on these flows.

In 1981 J. Lewowicz [73] studied the topological stability of the geodesic flow of a surface with non-positive curvature. As in Anosov's work, expansiveness is the key property. The definition of expansive flow used by Lewowicz can be found in [73, Lemma 4.1], there it is stated using local cross sections.

In 1991 R. O. Ruggiero in [114] proved that if the geodesic flow of a compact Riemannian manifold is C^1 persistently expansive then the closure of the set of periodic orbits is a hyperbolic set. If the manifold is two-dimensional then the geodesic flow is Anosov.

In 1993 M. Paternain [101] proved that expansive geodesic flows of compact Riemannian surfaces have no conjugate points. The proof relies on the construction of the stable foliation of [56, 100]. It is also shown that any two expansive geodesic flows on the same surface are topologically equivalent.

The interested reader should consult Ruggiero's survey [115] for more on expansive geodesic flows.

2.4.2 Homoclinic classes

The homoclinic class of a hyperbolic periodic point is the closure of the intersection of its stable manifold with its unstable manifold.

In 2005 [96] M. J. Pacifico, E. R. Pujals and J. L.Vieitez considered robustly expansive homoclinic classes of diffeomorphisms on three-dimensional manifolds. In this paper they proved that for an open and dense subset of the space of C^1 diffeomorphisms C^1-robustly expansive homoclinic classes are hyperbolic.

In 2009 M. J. Pacifico, E. R. Pujals, M. Sambarino and J. L.Vieitez [95] generalized the result in [96] to higher dimensions. They proved that robustly expansive codimension-one homoclinic classes are hyperbolic.

In 2008 M. J. Pacifico and J. L. Vieitez [97] considered robustly h-expansive homoclinic classes for surface diffeomorphisms. Recall that h-expansiveness (entropy expansiveness) was introduced by Bowen requiring that if a set has small diameter all the time then it has vanishing topological entropy. In cited paper it is proved that robustly h-expansive homoclinic classes have a dominated splitting, that is a weaker form of hyperbolicity. In 2010 [98] the same authors extended the previous result for arbitrary dimension. The converse result is also studied in these papers.

In 2013 T. Das, K. Lee and M. Lee [28] extended the previous and other problems to robustly cw-expansive homoclinic classes.

Chapter 3

The meaning of expansiveness

The expansiveness of a homeomorphism $f\colon X \to X$ of a compact metric space can be defined by requiring that the diagonal $\{(x, x) : x \in X\}$ be an isolated set for $f \times f$. Also continuum-wise expansiveness is related with isolated sets. Since many properties of expansiveness can be derived from results of isolated sets, let us start with this topic.

3.1 Isolated sets

Let $f\colon X \to X$ be a homeomorphism of a compact metric space.

Definitions 3.1.1. A subset $\Lambda \subset X$ is *f-isolated* if it is compact, invariant ($f(\Lambda) = \Lambda$) and there is an open set $U \subset X$ such that $\Lambda \subset U$ and $\cap_{n\in\mathbb{Z}} f^n(\operatorname{clos} U) = \Lambda$. In this case we say that U is an *isolating neighborhood*. If Λ is an isolating neighborhood of itself we say that Λ is *topologically isolated*. If $\cap_{n\geq 0} f^n(\operatorname{clos} U) = \Lambda$ we say that Λ is an *attractor* and if $\cap_{n\leq 0} f^n(\operatorname{clos} U) = \Lambda$ we say that Λ is a *repeller*.

Notice that Λ is f-isolated if and only if it is f^{-1}-isolated. Therefore, the following results holds for f^{-1} too.

Proposition 3.1.2. *Let Λ be an f-isolated set with isolating neighborhood U. If for some x it holds that $f^n(x) \in \operatorname{clos} U$ for all $n \geq 0$ then $\operatorname{dist}(f^n(x), \Lambda) \to 0$ as $n \to +\infty$.*

Proof. By contradiction assume that there are $\varepsilon > 0$ and an integer sequence $n_k \to +\infty$ such that if $y_k = f^{n_k}(x)$ then $\operatorname{dist}(y_k, \Lambda) \geq \varepsilon$ for all $k \geq 1$. Since $y_k \in \operatorname{clos} U$ and $\operatorname{clos} U$ is compact we can assume that $y_k \to y \in \operatorname{clos} U$. We have that if $|j| \leq n_k$ then $f^j(y_k) \in \operatorname{clos} U$. Since f is a homeomorphism we have that $f^j(y) \in \operatorname{clos} U$ for all $j \in \mathbb{Z}$. But this is a contradiction because $\operatorname{dist}(y, \Lambda) \geq \varepsilon$, in particular $y \notin \Lambda$, and $y \in \operatorname{clos} U$. □

Proposition 3.1.3. *If Λ is f-isolated but not topologically isolated then there is $x \notin \Lambda$ such that $\operatorname{dist}(f^n(x), \Lambda) \to 0$ as $n \to +\infty$ or $n \to -\infty$.*

Proof. Let U be an isolating neighborhood of Λ. By the previous proposition we have to find x such that $f^n(x) \in U$ for all $n \geq 0$ or for all $n \leq 0$. Assume that for all $x \in U$ there is $n \geq 0$ such that $f^n(x) \notin U$. Since Λ is not topologically isolated there is $x_k \to \Lambda$ with $x_k \notin \Lambda$. Then, there is $n_k \geq 0$ such that $f^j(x) \in U$ if $j = 0, 1, \dots, n_k - 1$ and $f^{n_k}(x_k) \notin U$. Suppose that $f^{n_k-1}(x_k) \to y \in \operatorname{clos}(U)$. We have that $y \notin \Lambda$ and since f is a homeomorphism we have that $f^n(y) \in \operatorname{clos}(U)$ for all $n \leq 0$. This finishes the proof. □

Proposition 3.1.4. *Let Λ be an f-isolated set with isolating neighborhood U. Then for all $\varepsilon > 0$ there is $n > 0$ such that if $\operatorname{dist}(x, \Lambda) > \varepsilon$ then there is $j \in \mathbb{Z}$ such that $|j| \leq n$ and $f^j(x) \notin U$.*

Proof. By contradiction suppose that there is $\varepsilon > 0$ such that for all $n > 0$ there is $x_n \in X$ such that $\operatorname{dist}(x_n, \Lambda) > \varepsilon$ and $f^j(x) \in U$ if $|j| \leq n$. Eventually taking a subsequence we can assume that $x_n \to x$. Then $\operatorname{dist}(x, \Lambda) \geq \varepsilon$ and it is easy to see that $f^n(x) \in \operatorname{clos}(U)$ for all $n \in \mathbb{Z}$. This contradicts that U is an isolating neighborhood of Λ. □

Proposition 3.1.5. *If Λ is a repeller f-isolated then for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\operatorname{dist}(x, \Lambda) < \delta$ then $\operatorname{dist}(f^{-j}(x), \Lambda) < \varepsilon$ for all $j \geq 0$.*

Proof. By contradiction assume that there are $\varepsilon > 0$, a sequence $x_n \in X$ and $j_n \geq 0$ such that $\operatorname{dist}(x_n, \Lambda) \to 0$ and $\operatorname{dist}(f^{-j_n}(x_n), \Lambda) \geq \varepsilon$. Let U be an isolating neighborhood of Λ and consider $\sigma \in (0, \varepsilon)$ such that $B_\sigma(\Lambda) \subset U$. Let i_n be such that $\operatorname{dist}(f^{-i_n}(x_n), \Lambda) \geq \sigma$ and $\operatorname{dist}(f^{-i}(x_n), \Lambda) < \sigma$ for all $i = 0, 1, \dots, i_n - 1$. Now a limit point of $f^{-i_n}(x_n)$ contradicts that Λ is a repeller. □

3.2 Positive expansiveness

If the separation is required to be in positive time we have positive expansiveness.

Definition 3.2.1. A homeomorphism $f \colon X \to X$ is said to be *positive expansive* if there exists $\delta > 0$ such that if $\operatorname{dist}(f^n(x), f^n(y)) < \delta$ for all $n \geq 0$ then $x = y$.

Remark 3.2.2. *Consider $g \colon X \times X \to X \times X$ defined by $g(x, y) = (f(x), f(y))$. Define the diagonal $\Delta = \{(x, x) : x \in X\}$. Notice that f is positive expansive if and only if Δ is a repeller g-isolated set. Therefore, applying Proposition 3.1.5, for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\operatorname{dist}(x, y) \leq \delta$ then $\operatorname{dist}(f^{-n}(x), f^{-n}(y)) \leq \varepsilon$ for all $n \geq 0$.*

This concept is interesting in the study of the dynamics of endomorphisms i.e. continuous maps not necessarily injective. In the case of homeomorphisms we will show that the study of positive expansive homeomorphisms is reduced to permutations on finite sets. We follow the proof from [27].

Theorem 3.2.3 (Schwartzman [118]). *The only compact metric spaces admitting positive expansive homeomorphisms are finite sets.*

Proof. Fix an expansive constant ε and the corresponding δ from Remark 3.2.2. Now cover X by finitely many open sets $U_1, \dots, U_N$ with diameter smaller than ε. If X contains more then N points, consider $A \subset X$ such that $|A| = N + 1$. For every $k \geq 0$, there are two different points $x_k, y_k \in A$ such that $f^k(x_k)$ and $f^k(y_k)$ lie in the same set U_{n_k} of the covering. Then $\text{dist}(f^i(x_k), f^i(y_k)) \leq \delta$ for $i \leq k$. Since $A \times A$ is finite, there exist two points $x, y \in A$ such that $\text{dist}(f^i(x), f^i(y)) \leq \delta$ for infinitely many values of $i \geq 0$. By the previous argument, $\text{dist}(f^i(x), f^i(y)) \leq \delta$ for all $i \geq 0$. This contradicts positive expansiveness. □

This result was first proved in [118] and another proof can be found in [75]. In Chapter 7 we will consider positive expansive flows. Theorem 3.2.3 means that for invertible dynamics one has to allow the separation to occur at positive or negative times. In this way we have expansiveness.

3.3 Expansiveness

As before, consider $f\colon X \to X$ a homeomorphism of a compact metric space.

Definition 3.3.1. We say that f is *expansive* if there is a constant $\delta > 0$ such that if $\text{dist}(f^n(x), f^n(y)) \leq \delta$ for all $n \in \mathbb{Z}$ then $x = y$. In this case we say that δ is an *expansive constant.*

The following equivalent definition was used by Utz in [128]. Denote by $g\colon X \times X \to X \times X$ the homeomorphism induced by f by $g(x, y) = (f(x), f(y))$.

Proposition 3.3.2. *A homeomorphism f is expansive if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is a g-isolated set.*

Proof. It follows by the definitions. □

Proposition 3.3.3. *If δ is an expansive constant for f and* $\text{dist}(f^n(a), f^n(b)) \leq \delta$ *for all $n \geq 0$ then* $\text{dist}(f^n(a), f^n(b)) \to 0$ *as $n \to +\infty$.*

Proof. It follows by Proposition 3.1.2. □

Remark 3.3.4. *In* [128] *Utz proved that if $f\colon X \to X$ is an expansive homeomorphism and $|X| = \infty$ then there are two points whose orbits are asymptotic in at least one sense. It is a consequence of Proposition 3.1.3. The next proposition is a stronger result.*

Proposition 3.3.5. *If X is infinite and $f\colon X \to X$ is an expansive homeomorphism then there are $a, b, c, d \in X$ such that $a \neq b$, $c \neq d$ and*

$$\lim_{n\to+\infty} \operatorname{dist}(f^n(a), f^n(b)) = \lim_{n\to-\infty} \operatorname{dist}(f^n(c), f^n(d)) = 0.$$

Proof. Let δ be an expansive constant for f. By Theorem 3.2.3 we have that f is not positive expansive. Therefore, there are $a, b \in X$ such that $\operatorname{dist}(f^n(a), f^n(b)) \leq \delta$ for all $n \geq 0$. So, we conclude by Proposition 3.3.3. To obtain the points c, d we can argue using that f^{-1} is not positive expansive. □

Two different points $a, b \in X$ are *doubly-asymptotic* if $\operatorname{dist}(f^n(a), f^n(b)) \to 0$ as $n \to +\infty$ and $n \to -\infty$. As we will see in Section 3.7.1 there are expansive homeomorphisms without doubly-asymptotic points. We will also see in Section 3.7.1 that there can be trajectories without asymptotic points.

3.3.1 Uniform expansiveness

Let $\delta > 0$ be an expansive constant for $f\colon X \to X$ and consider $N\colon X \times X \to \mathbb{N} \cup \{\infty\}$ as the function defined by

$$N(x, y) = \begin{cases} \infty & \text{if } x = y, \\ \min\{|n| : \operatorname{dist}(f^n x, f^n y) > \delta, n \in \mathbb{Z}\} & \text{if } x \neq y. \end{cases} \tag{3.1}$$

Definition 3.3.6. A homeomorphism is *uniformly expansive* if for all $\sigma > 0$ there exists $m > 0$ such that if $\operatorname{dist}(x, y) > \sigma$ then $N(x, y) \leq m$.

Proposition 3.3.7 (Bryant [22])**.** *If X is a compact metric space then every expansive homeomorphism is uniformly expansive.*

Proof. It follows by Proposition 3.1.4. □

3.4 Variations of expansiveness

3.4.1 Point-wise expansiveness

The next definition is associated with a variable expansive constant.

Definition 3.4.1. A homeomorphism f is said to be *pointwise expansive* if for each $x \in X$ there is $\delta(x) > 0$ such that if $\operatorname{dist}(f^n(x), f^n(y)) < \delta(x)$ for all $n \in \mathbb{Z}$ then $x = y$.

If the function δ is continuous we have expansiveness since X is compact. In 1970 Reddy [107] introduced this definition and presented an example of a pointwise expansive homeomorphism that is not expansive. In Example 3.4.10 below we present a variation of this example, showing that in fact pointwise-expansiveness does not imply cw-expansiveness (see Definition 3.4.5).

In Section 3.7.1 we show that there are non-trivial positive pointwise-expansive homeomorphisms. It is a minimal system on a Cantor set.

Proposition 3.4.2 (Utz [128], Reddy [107]). *Let $p \in X$ be an accumulation point and assume that p is a periodic point of the pointwise expansive homeomorphism f. Then there is x such that* $\operatorname{dist}(f^n(x), f^n(p)) \to 0$ *as $n \to +\infty$ or $n \to -\infty$.*

Proof. Pointwise expansiveness implies that periodic orbits of f are f-isolated sets. Therefore, the result follows by Proposition 3.1.3. □

Definition 3.4.3. We say that p is a *stable point* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\operatorname{dist}(x, p) < \delta$ then $\operatorname{dist}(\phi_t(x), \phi_t(p)) < \varepsilon$ for all $t \geq 0$.

Remark 3.4.4. *If f is point-wise expansive and p is a stable periodic point then the orbit of p is an attractor f-isolated set.*

Some techniques of expansive homeomorphisms involving arcs or connected sets seems not to be adaptable for point-wise expansiveness. But another weaker version of expansiveness exist.

3.4.2 Continuum-wise expansiveness

A *continuum* is a compact connected metric space. Every singleton $\{x\}$ is a continuum. A continuum is *non-trivial* if it is not a singleton. If a homeomorphism $f\colon X \to X$ is expansive and $C \subset X$ is a non-trivial continuum then there is $n \in \mathbb{Z}$ such that $\operatorname{diam}(f^n C) > \delta$ if $\delta > 0$ is an expansive constant for f. This property is very important in the study of expansiveness.

Definition 3.4.5. We say that f is *continuum-wise expansive* if there is $\delta > 0$ such that if $C \subset X$ is a continuum such that $\operatorname{diam}(f^n(C)) < \delta$ for all $n \in \mathbb{Z}$ then C is a singleton.

In 1993 Kato in [61] introduced this definition, weaker than expansiveness, for which a great number of techniques and results of expansive homeomorphisms were adapted.

Remark 3.4.6. *On totally disconnected spaces, as a Cantor set, every homeomorphism is cw-expansive. This is because there are no non-trivial continua. Therefore cw-expansiveness does not imply point-wise expansiveness (consider for example, the identity of a Cantor set). An example on a compact surface is presented in Section 5.3.*

Let $\mathcal{C}$ denote the space of continua subsets of X with the Hausdorff metric. We recall the definition.

Definition 3.4.7. If K, L are compact subsets of X then the Hausdorff distance is

$$\operatorname{dist}_H(K, L) = \inf\{\varepsilon > 0 : K \subset B_\varepsilon(L), L \subset B_\varepsilon(K)\}.$$

Denote by $f_c\colon \mathcal{C} \to \mathcal{C}$ the homeomorphism induced by a homeomorphism $f\colon X \to X$ by $f_c(A) = \{f(x) : x \in A\}$. Define $F_1 = \{\{x\} \in \mathcal{C} : x \in X\}$ the space of singletons of X. Notice that F_1 is compact and invariant under f_c.

Proposition 3.4.8. *A homeomorphism f is cw-expansive if and only if F_1 is an f_c-isolated set.*

Proof. It follows by definitions. □

Proposition 3.4.9 (Mañé [78])**.** *If $f\colon X \to X$ is a cw-expansive homeomorphism and X is not totally disconnected there is a non-trivial continuum C such that* $\operatorname{diam}(f^n C) \to 0$ *as $n \to +\infty$ or $n \to -\infty$.*

Proof. It is a consequence of Proposition 3.1.3. □

Example 3.4.10. *Let us sketch the construction of a Peano continuum admitting a pw-expansive homeomorphism that is not cw-expansive. Let $f\colon S \to S$ be an expansive homeomorphism of a two-dimensional torus (that we can consider embedded in Euclidean $\mathbb{R}^3$). Denote by $x_1, x_2, x_3, \dots$ a sequence of periodic points of f such that if p_i is the period of x_i then the sequence p_i is increasing. For each $f^j(x_i)$ with $j = 1, \dots, p_i$ consider a torus S_{ij} of diameter smaller than $1/p_i$ such that $S_{ij} \cap S_{i'j'} \neq \emptyset$ only if $i = i'$ and $j = j'$. Moreover, assume that $S \cap S_{ij} = \{f^j(x_i)\}$. Consider the space $X = S \cup \cup S_{ij}$ the union of all these tori. On X consider a homeomorphism g such that $g|S = f$ and $g(S_{ij}) = S_{i(j+1)}$, $g(S_{ip_i}) = S_{i1}$ and $g^{p_i}\colon S_{ij} \to S_{ij}$ is an expansive homeomorphism. We have that the diameter of the tori $g^n(S_{ij})$ is small for all $n \in \mathbb{Z}$ and then g is not cw-expansive. By construction it can be proved that g is pw-expansive.*

3.4.3 N-expansiveness

Let us give another notion that was introduced in [87]. The cardinality of a set A will be denoted as $|A|$.

Definition 3.4.11. Given $N \geq 1$, a homeomorphism is N-*expansive* if there is $\delta > 0$ such that if $\operatorname{diam}(f^k(A)) < \delta$ for all $k \in \mathbb{Z}$ then $|A| \leq N$.

By definitions we have that expansiveness implies N-expansiveness. Also N-expansiveness implies pw-expansiveness and cw-expansiveness. See Morales's paper [87] for more properties of N-expansive homeomorphisms.

3.5 Hyperbolic metric

Hyperbolic sets are known to be expansive. The converse is, in some sense, true. We mean, every expansive homeomorphism admits a hyperbolic metric. This is the topic of the present section. The construction is due to Fathi [30]. Let $f\colon X \to X$ be an expansive homeomorphism on the compact metric space (X, dist).

Definition 3.5.1. A metric d on X defining the same topology as dist is *hyperbolic* if there are numbers $k > 1$ and $\delta > 0$ such that

$$\max\{d(f(x), f(y)), d(f^{-1}(x), f^{-1}(y))\} \geq \min\{kd(x, y), \delta\}$$

for all $x, y \in X$. In this case we say that k is the *expanding factor*.

Since f and f^{-1} are continuous, there is $\sigma > 0$ such that

$$\max\{d(fx, fy), d(f^{-1}x, f^{-1}y)\} < \delta$$

if $\operatorname{dist}(x, y) < \sigma$. So, the hyperbolicity of dist implies that if $\operatorname{dist}(x, y) < \sigma$ then $\operatorname{dist}(fx, fy) \geq k \operatorname{dist}(x, y)$ or $\operatorname{dist}(f^{-1}x, f^{-1}y) \geq k \operatorname{dist}(x, y)$. With a hyperbolic metric one has a nice control of nearby points: the distance exponentially increases in one positive or negative iterate.

To show how useful a hyperbolic metric is, we offer the following result.

Proposition 3.5.2. *If* dist *is a* (k, δ)*-hyperbolic metric for* f *and* $\operatorname{dist}(f^n x, f^n y) < \delta$ *for all* $n \geq 0$ *then*

$$\operatorname{dist}(fx, fy) \leq \frac{\operatorname{dist}(x, y)}{k}.$$

Moreover,

$$\operatorname{dist}(f^i x, f^i y) \leq \frac{\operatorname{dist}(x, y)}{k^i}$$

for all $i \geq 0$.

Proof. Let $x_i = f^i x$ and $y_i = f^i y$. By contradiction suppose that $k \operatorname{dist}(x_1, y_1) > \operatorname{dist}(x_0, y_0)$. The hyperbolicity of the metric requires that

$$\max\{\operatorname{dist}(x_2, y_2), \operatorname{dist}(x_0, y_0)\} \geq \min\{k \operatorname{dist}(x_1, y_1), \delta\}.$$

Since $\operatorname{dist}(x_0, y_0) < \delta$ and $\operatorname{dist}(x_2, y_2) < \delta$ we have that $\operatorname{dist}(x_2, y_2) \geq k \operatorname{dist}(x_1, y_1)$. Again

$$\max\{\operatorname{dist}(x_3, y_3), \operatorname{dist}(x_1, y_1)\} \geq \min\{k \operatorname{dist}(x_2, y_2), \delta\}.$$

Similar argument gives us that $\operatorname{dist}(x_3, y_3) \geq k \operatorname{dist}(x_2, y_2)$. By induction we can prove that

$$\operatorname{dist}(x_{i+1}, y_{i+1}) \geq k \operatorname{dist}(x_i, y_i),$$

for all $i \geq 0$. This contradicts that $\operatorname{dist}(x_i, y_i) < \delta$ for all $i \geq 0$. □

3.5.1 Construction of a hyperbolic metric

Let $\delta > 0$ be an expansive constant for f. By Proposition 3.3.7 there is m such that if $\text{dist}(x, y) > \delta/2$ then $\max_{|n|\leq m} \text{dist}(f^n x, f^n y) > \delta$. Let $\alpha > 1$ be such that $\alpha^m \leq 2$. Recall that the function N was defined as

$$N(x,y) = \begin{cases} \infty & \text{if } x = y, \\ \min\{|n| : \text{dist}(f^n x, f^n y) > \delta, n \in \mathbb{Z}\} & \text{if } x \neq y. \end{cases} \tag{3.2}$$

and define $\rho\colon X \times X \to \mathbb{R}$ by

$$\rho(x,y) = \alpha^{-N(x,y)}.$$

We will show that ρ is a quasi-metric. So applying Proposition A.1 we have the metric D associated to ρ. Choose n_0 such that $K = (\alpha^{n_0}/4) > 1$. Let $k = K^{1/n_0}$. We define another metric d by

$$d(x,y) = \max_{|i|\leq n_0 - 1} \frac{D(f^i(x), f^i(y))}{k^{|i|}}. \tag{3.3}$$

We will show that d is a hyperbolic metric for the expansive homeomorphism f.

To show that ρ is a quasi-metric we need some Lemmas.

Lemma 3.5.3. *If*

$$\max_{|i|\leq n-1} \rho(f^i(x), f^i(y)) \leq \frac{1}{\alpha}$$

then

$$\max\{\rho(f^n(x), f^n(y)), \rho(f^{-n}(x), f^{-n}(y))\} \geq \alpha^n \rho(x,y). \tag{3.4}$$

Proof. Notice that the hypothesis means $N(x, y) \geq n$ and the thesis is equivalent to

$$N(x,y) \geq n + \min\{N(f^{-n}(x), f^{-n}(y)), N(f^n(x), f^n(y))\}.$$

So it is trivial. □

Lemma 3.5.4. *For all $x, y, z \in X$ it holds that*

$$\min\{N(z,y), N(z,x)\} \leq m + N(x,y).$$

Proof. This follows from the triangular inequality for the metric dist and the definition of m. The idea is: when the iterates of x and y are at a distance greater then δ then z can not be at a distance less than $\delta/2$ from x and y. □

Proposition 3.5.5. *The function ρ is a quasi-metric defining the topology of* dist.

Proof. The first three items (of the definition of quasi-metric) follows by definitions. To prove the last one recall that $\alpha^m \leq 2$ and the previous Lemma.

The quasi-metric ρ defines the topology of dist, because f is an expansive homeomorphisms. □

From (3.4) and (A.10), we have that if

$$\max_{|i|\leq n-1} D(f^i(x), f^i(y)) \leq \frac{1}{4\alpha}$$

then

$$\max\{D(f^n(x), f^n(y)), D(f^{-n}(x), f^{-n}(y))\} \geq \frac{\alpha^n}{4} D(x, y). \tag{3.5}$$

Theorem 3.5.6 (Fathi [30]). *Expansive homeomorphisms on compact metric spaces admit hyperbolic metrics.*

Proof. By direct inspection, it is easy to establish the following inequality:

$$\max\{d(f(x), f(y)), d(f^{-1}(x), f^{-1}(y))\} \geq \max_{0<|i|\leq n_0} \frac{D(f^i(x), f^i(y))}{k^{|i|-1}}.$$

Now this last quantity is the maximum of the following two quantities A and B:

$$A = \max_{0<|i|<n_0} \frac{D(f^i(x), f^i(y))}{k^{|i|-1}} = k \max_{0<|i|<n_0} \frac{D(f^i(x), f^i(y))}{k^{|i|}}, \tag{3.6}$$

and

$$B = \frac{\max\{D(f^{n_0}(x), f^{n_0}(y)), D(f^{-n_0}(x), f^{-n_0}(y))\}}{k^{n_0}}.$$

Suppose now that $d(x, y) < \frac{1}{4\alpha k^{n_0-1}}$. Then by (3.5), (3.3) and the definition of k, we get:

$$B \geq kD(x, y). \tag{3.7}$$

It is easy to conclude from (3.6) and (3.7) that if $d(x, y) \geq 1/4\alpha k^{n_0-1}$ then

$$\max\{d(f(x), f(y)), d(f^{-1}(x), f^{-1}(y))\} \geq kd(x, y). \tag{3.8}$$

Since X is compact, we can find $\delta > 0$ such that if $d(x, y) \geq 1/4\alpha k^{n_0-1}$ then

$$\max\{d(f(x), f(y)), d(f^{-1}(x), f^{-1}(y))\} \geq \delta. \tag{3.9}$$

From (3.8) and (3.9), we have

$$\max\{d(f(x), f(y)), d(f^{-1}(x), f^{-1}(y))\} \geq \min\{kd(x, y), \delta\}.$$

for all $x, y \in X$. □

3.6 Lyapunov functions

As we have explained in the previous section, expansiveness is equivalent with the existence of a hyperbolic metric. Another characterization can be obtained using Lyapunov functions. This idea is due to Lewowicz. In this section we develop the technique of Lyapunov functions for expansive systems from a different viewpoint from Lewowicz's one. Our method is based on isolated sets for flows in the sense of Conley.

In Dynamical Systems and Differential Equations it is important to determine the stability of trajectories and a well known technique for this purpose is to find a Lyapunov function. In order to fix ideas consider a continuous flow $\phi\colon \mathbb{R}\times X \to X$ on a compact metric space (X, dist) with a singular (or equilibrium) point $p \in X$, i.e., $\phi_t(p) = p$ for all $t \in \mathbb{R}$. A *Lyapunov function* for p is a continuous non-negative function that vanishes only at p and strictly decreases along the orbits close to p.

Definition 3.6.1. We say that p is *asymptotically stable* if it is stable and there is $\delta_0 > 0$ such that if $\mathrm{dist}(x, p) < \delta_0$ then $\phi_t(x) \to p$ as $t \to +\infty$.

The existence of a Lyapunov function for an equilibrium point implies the asymptotic stability of the equilibrium point.

A remarkable result, first proved by Massera in [79], is the converse: every asymptotically stable singular point admits a Lyapunov function of class C^1. Later, other authors obtained Lyapunov functions with different methods, see for example [14, 26]. In [55] a generalization is proved in the context of arbitrary metric spaces. The purpose of this section is to develop a new technique that allows us to construct Lyapunov functions for different dynamical systems as: isolated sets, expansive homeomorphisms and continuum-wise expansive homeomorphisms. Our techniques are based on the size function μ introduced by Whitney in [135].

In order to give a motivation let us show how to construct a Lyapunov function for an asymptotically stable singular point. As before, denote by $\mathbb{K}$ the hyper-space of non-empty compact subsets of X with the Hausdorff distance.

Definition 3.6.2. A *size function* is a continuous function $\mu\colon \mathbb{K} \to \mathbb{R}$ satisfying:

1. $\mu(A) \geq 0$ with equality if and only if A has only one point,
2. if $A \subset B$ and $A \neq B$ then $\mu(A) < \mu(B)$.

In [135] it is proved that size functions exists for every compact metric space.

Theorem 3.6.3 (Massera [79])**.** *If ϕ is a continuous flow on X with an asymptotically stable singular point p then there are an open set U containing p and a continuous function $V : U \to \mathbb{R}$ satisfying:*

1. *$V(x) \geq 0$ for all $x \in U$ with equality if and only if $x = p$ and*

2. *if* $t > 0$ *and* $\{\phi_s(x) : s \in [0,t]\} \subset U$ *then* $V(\phi_t(x)) < V(x)$.

Proof. By the conditions on p there are $\delta_0, \delta > 0$ such that if $\text{dist}(x,p) < \delta$ then $\phi_t(x) \in B_{\delta_0}(p)$ for all $t \geq 0$ and $\phi_t(x) \to p$ as $t \to \infty$. Define $U = B_\delta(p)$ and $V : U \to \mathbb{R}$ as

$$V(x) = \mu(\{\phi_t(x) : t \geq 0\} \cup \{p\})$$

where μ is a size function. Since $\phi_t(x) \to p$ we have that

$$O(x) = \{\phi_t(x) : t \geq 0\} \cup \{p\} \tag{3.10}$$

is a compact set for all $x \in U$. Notice that if $t > 0$ then $O(\phi_t(x)) \subset O(x)$ and the inclusion is proper. Therefore, $V(\phi_t(x)) < V(x)$ because μ is a size function. Also notice that $V(p) = 0$ and $V(x) > 0$ if $x \neq p$. In order to prove the continuity of V, we will prove the continuity of $O : U \to K(X)$, the map defined by (3.10). Since μ is continuous we will conclude the continuity of V.

Let us prove the continuity of O at $x \in U$. Take $\varepsilon > 0$. By the asymptotic stability of p there are $\rho, T > 0$ such that if $y \in B_\rho(x)$ then $\phi_t(y) \in B_{\varepsilon/2}(p)$ for all $t \geq T$. By the continuity of the flow, there is $r > 0$ such that if $y \in B_r(x)$ then $\text{dist}(\phi_t(x), \phi_t(y)) < \varepsilon$ for all $t \in [0,T]$. Now it is easy to see that if $y \in B_{\min\{\rho,r\}}(x)$ then $\text{dist}_H(O(x), O(y)) < \varepsilon$, proving the continuity of O at x and consequently the continuity of V. □

Let us recall that size functions can be easily defined. A variation of the construction given in [135], adapted for compact metric spaces, is the following. Let $q_1, q_2, q_3, \dots$ be a sequence dense in X. Define $\mu_i : \mathbb{K} \to \mathbb{R}$ as

$$\mu_i(A) = \max_{x \in A} \text{dist}(q_i, x) - \min_{x \in A} \text{dist}(q_i, x).$$

The following formula defines a size function $\mu : \mathbb{K} \to \mathbb{R}$

$$\mu(A) = \sum_{i=1}^{\infty} \frac{\mu_i(A)}{2^i},$$

as proved in [135]. In Section 3.6.2 we extend Theorem 3.6.3 by constructing a Lyapunov function for an isolated invariant sets.

For the study of expansive homeomorphisms Lewowicz introduced in [72] Lyapunov functions. He proved that expansiveness is equivalent with the existence of such function. In Section 3.6.3 we give a different proof of this result by constructing a Lyapunov function defined for compact subsets of the space. With our techniques we prove that continuum-wise expansiveness is equivalent with the existence of a Lyapunov function on continua subsets of the space.

Let us start explaining how to construct a Lyapunov function in $X \times X$ using a hyperbolic metric.

3.6.1 Lyapunov functions via hyperbolic metrics

In [75] a quadratic form is constructed for an Anosov diffeomorphism. In this section we will use the technique of [75] to construct a Lyapunov function from a hyperbolic metric.

A *Lyapunov function* for a homeomorphisms $f\colon X \to X$ is a continuous map $V\colon N \to \mathbb{R}$ defined on a compact neighborhood of the diagonal on $X \times X$, such that $V(x,x) = 0$ for all $x \in X$ and

$$\Delta V(x,y) = V(fx,fy) - V(x,y) > 0$$

if $x \neq y$ and $x, y, fx, fy \in N$.

Theorem 3.6.4 (Lewowicz [72]). *A homeomorphism of a compact metric space is expansive if and only if it admits a Lyapunov function.*

Proof. Converse. Let $\alpha > 0$ be such that if $\operatorname{dist}(x,y) < \alpha$ then $(x,y) \in N$. We will show that α is an expansive constant. Suppose that $\operatorname{dist}(f^n x, f^n y) < \alpha$ for all $n \in Z$. If $V(x,y) > 0$ then $V(f^n x, f^n y) > V(x,y)$ for all $n > 0$. Then there is $\rho > 0$ such that $\operatorname{dist}(f^n x, f^n y) > \rho$ for all $n \geq 0$ because V is continuous and vanishes on the diagonal. We have that

$$\min\{\Delta V(x,y) : \operatorname{dist}(x,y) \geq \rho\} = \mu$$

is positive since N is compact. So

$$V(f^n x, f^n y) = V(x,y) + \sum_{i=0}^{n-1} \Delta V(f^i x, f^i y) \geq V(x,y) + (n-1)\mu,$$

which is a contradiction because V is bounded. If $V(x,y) < 0$ we get the same contradiction considering f^{-1}. If $V(x,y) = 0$ and $x \neq y$ then $V(fx,fy) > 0$ and we can repeat the argument with fx and fy. We have proved that $x = y$.

Direct. Now we assume that f is expansive. We consider a (k,δ)-hyperbolic metric dist. If $k^m > 2$ then there is $\sigma > 0$ such that if $0 < \operatorname{dist}(x,y) \leq \sigma$ then

$$\max\{\operatorname{dist}(f^m x, f^m y), \operatorname{dist}(f^{-m} x, f^{-m} y)\} > 2\operatorname{dist}(x,y). \tag{3.11}$$

Let $N = \{(x,y) \in X \times X : \operatorname{dist}(x,y) \leq \sigma\}$ and define $V\colon N \to \mathbb{R}$ as

$$V(x,y) = \sum_{i=0}^{m-1} \operatorname{dist}(f^{m+i} x, f^{m+i} y) - \sum_{i=0}^{m-1} \operatorname{dist}(f^i x, f^i y).$$

Notice that

$$\Delta V(x,y) = \operatorname{dist}(f^{2m} x, f^{2m} y) - 2\operatorname{dist}(f^m x, f^m y) + \operatorname{dist}(x,y).$$

So, applying (3.11) we have that $\Delta V(x,y) > 0$ if $x \neq y$. □

3.6.2 Lyapunov Functions for Isolated Sets

In this section we consider continuous flows on compact metric spaces. The purpose is to construct a Lyapunov for an isolated set of the flow using a size function. First we consider the case of an isolated set consisting of a point.

Isolated Singularities Let ϕ be a continuous flow on a compact metric space (X, dist). A point $p \in X$ is *singular* for ϕ if $\phi_t(p) = p$ for all $t \in \mathbb{R}$. A singular point $p \in X$ is *isolated* if there is an open *isolating neighborhood* U of p such that if $\phi_{\mathbb{R}}(x) \subset U$ then $x = p$.

Definition 3.6.5. An open set U is an *adapted neighborhood* of an isolated singular point $p \in U$ if for every orbit segment $l \subset \mathrm{clos}(U)$ with extreme points in U it holds that $l \subset U$.

Given a set $A \subset X$ and $x \in A$ denote by $\mathrm{comp}_x(A)$ the connected component of A that contains the point x.

Proposition 3.6.6. *Every isolated singular point has an adapted neighborhood.*

Proof. Let $r > 0$ be such that $\mathrm{clos}(B_r(p)) \subset N$. For $\rho \in (0, r)$ define the set

$$U_\rho = \{x \in B_r(p) : \mathrm{comp}_x(\phi_{\mathbb{R}}(x) \cap B_r(p)) \cap B_\rho(p) \neq \emptyset\}.$$

By the continuity of ϕ we have that U_ρ is an open set for all $\rho \in (0, r)$. Let us prove that if ρ is sufficiently small then U_ρ is an adapted neighborhood. By contradiction, suppose that there are $\rho_n \to 0$, $a_n, b_n \in U_{\rho_n}$, $t_n \geq 0$ such that $b_n = \phi_{t_n}(a_n)$ and $l_n = \phi_{[0,t_n]}(a_n) \subset \mathrm{clos}(U_{\rho_n})$ but l_n is not contained in U_{ρ_n}.

If $l_n \subset B_r(p)$ then l_n would be contained in U_{ρ_n}. Since we know that this is not the case there is $s_n \in (0, t_n)$ such that $c_n = \phi_{s_n}(a_n) \in \partial B_r(p)$. Since $a_n, b_n \in U_{\rho_n}$ we know that $\mathrm{comp}_{a_n}(\phi_{\mathbb{R}}(a_n) \cap B_r(p)) \cap B_\rho(p) \neq \emptyset$ and $\mathrm{comp}_{b_n}(\phi_{\mathbb{R}}(b_n) \cap B_r(p)) \cap B_\rho(p) \neq \emptyset$. Then, there must be $u_n < 0$ and $v_n > 0$ such that $\phi_{u_n}(c_n), \phi_{v_n}(c_n) \in B_{\rho_n}(p)$ with $\phi_{[u_n,v_n]}(c_n) \subset \mathrm{clos}(B_r(p)$, $u_n \to -\infty$ and $v_n \to +\infty$. If c is a limit point of c_n we have that $\phi_{\mathbb{R}}(c) \subset B_r(p)$ and $c \neq p$. This contradicts that $\mathrm{clos}(B_r(p))$ is contained in an insolating neighborhood of p and proves the result. □

Fix an isolated point p with an adapted neighborhood U. Consider the sets

$$W^s_U(p) = \{x \in U : \lim_{t\to+\infty} \phi_t(x) = p \text{ and } \phi_{\mathbb{R}^+}(x) \subset U\},$$
$$W^u_U(p) = \{x \in U : \lim_{t\to-\infty} \phi_t(x) = p \text{ and } \phi_{\mathbb{R}^-}(x) \subset U\},$$

.

For $x \in U$ define the orbit segments

$$O^+_U(x) = \mathrm{comp}_x(U \cap \phi_{[0,+\infty)}(x)),$$
$$O^-_U(x) = \mathrm{comp}_x(U \cap \phi_{(-\infty,0]}(x)).$$

Define $C = X \setminus U$ and let $V_p^+, V_p^- \colon U \to \mathbb{K}$ be defined as

$$\begin{cases} V_p^+(x) = \operatorname{clos}(O_U^+(x) \cup W_U^u(p)) \cup C, \\ V_p^-(x) = \operatorname{clos}(O_U^-(x) \cup W_U^s(p)) \cup C. \end{cases}$$

Definition 3.6.7. A *Lyapunov function* for an isolated point p is a continuous function $V \colon U \to \mathbb{R}$ defined in a neighborhood of p such that if $t > 0$ and $\phi_{[0,t]}(x) \subset U \setminus \{p\}$ then $V(x) > V(\phi_t(x))$.

The following is a well known result, see for example [26]. The proof via size functions seems to be a new one.

Theorem 3.6.8. *If p is an isolated point and U is an adapted neighborhood of p then the maps V_p^+ and V_p^- are continuous in U. If in addition, μ is a size function on $\mathbb{K}$ then $V \colon U \to \mathbb{R}$ defined as*

$$V(x) = \mu(V_p^+(x)) - \mu(V_p^-(x))$$

is a Lyapunov function for p.

Proof. Let us prove the continuity of V_p^+ by contradiction. Assume that $x_n \to x \in U$ and $V_p^+(x_n) \to K$ with the Hausdorff distance but $K \neq V_p^+(x)$. By definitions we have that

$$\operatorname{clos}(W_U^u(p)) \cup C \subset K \cap V_p^+(x). \tag{3.12}$$

Recall that C was defined as the complement of U in X. Take a point $y \in K \setminus V_p^+(x) \cup V_p^+(x) \setminus K$. By the inclusion (3.12) we know that $y \notin \operatorname{clos}(W_U^u(p)) \cup C$. We divide the proof in two cases.

Case 1. Suppose first that $y \in K \setminus V_U^+(x)$. Since $y \in K$ there is a sequence $t_n \geq 0$ such that $\phi_{t_n}(x_n) \to y$ and $\phi_{[0,t_n]}(x_n) \subset U$. If $t_n \to \infty$ then $x \in W_U^s(p)$. Consequently, $y \in W_U^u(p)$, which is a contradiction. Therefore t_n is bounded. Without loss of generality assume that $t_n \to \tau \geq 0$ and then $\phi_\tau(x) = y$. Thus $\phi_{[0,\tau]}(x) \subset \operatorname{clos}(U)$. Since $y \notin C$ we have that $y \in U$. Now, since U is an adapted neighborhood we conclude that $\phi_{[0,\tau]}(x) \subset U$ and then $y \in O^+(x) \subset V_p^+(x)$. This contradiction finishes this case.

Case 2. Now assume that $y \in V_p^+(x) \setminus K$. In this case we have that $y = \phi_s(x)$ for some $s \geq 0$ and $\phi_{[0,s]}(x) \subset U$. Then $\phi_s(x_n) \to y$ and $y \in K$. This contradiction proves that V_p^+ is continuous in U.

The continuity of V_p^- is proved in a similar way. Let us show that V is a Lyapunov function for p. The continuity of V in U follows by the continuity of V_p^+, V_p^- and the size function μ.

Now take $x \notin U \setminus \{p\}$. We will show that V decreases along the orbit segment of x contained in U. Notice that for all $t > 0$, $O_U^+(\phi_t(x)) \subset O_U^+(x)$ if $\phi_{[0,t]}(x) \subset U$. Therefore $V_p^+(\phi_t(x)) \leq V_p^+(O^+(x))$. The equality can only hold if $x \in W_U^u(p)$. But in this case we have that $x \notin W_U^s(p)$ because $W_U^u(p) \cap W_U^s(p) = \{p\}$. Then $V_p^-(\phi_t(x)) > V_p^-(x)$. Therefore, $V(\phi_t(x)) < V(x)$ and V is a Lyapunov function for p. □

Isolated Sets Let $\phi\colon \mathbb{R}\times X \to X$ be a continuous flow on a compact metric space X. Consider a ϕ-invariant set $\Lambda \subset X$, i.e., $\phi_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. We say that Λ is an *isolated set* with *isolating neighborhood* U if $\phi_{\mathbb{R}}(x) \subset U$ implies $x \in \Lambda$.

Definition 3.6.9. A *Lyapunov function* for an isolated set Λ is a continuous function $V : U \to \mathbb{R}$ defined on an open set U containing Λ such that:

1. $V(x) = 0$ if $x \in \Lambda$,
2. if $\phi_{[0,t]}(x) \subset U \setminus \Lambda$ then $V(x) > V(\phi_t(x))$.

Let us show how the construction of a Lyapunov function for an isolated set can be reduced to the case of an isolated singular point.

Theorem 3.6.10. *Every isolated set admits a Lyapunov function.*

Proof. Consider the set $Y = (X \setminus \Lambda) \cup \{\Lambda\}$. On Y define the distance d as

$$\mathrm{d}(x, y) = \min\{\mathrm{dist}(x, y), \mathrm{dist}(x, \Lambda) + \mathrm{dist}(y, \Lambda)\}.$$

It is easy to see that (Y, d) is a compact metric space. Also, the flow ϕ induces naturally a flow ϕ' on Y with Λ as an isolated singular point. Consider from Theorem 3.6.8 a Lyapunov function for Λ as an isolated singular point of ϕ'. This function naturally defines a Lyapunov function for Λ as an isolated set of ϕ. □

3.6.3 Applications to expansive homeomorphisms

Let $f\colon X \to X$ be a homeomorphism of a compact metric space (X, dist).

Theorem 3.6.11. *Every isolated set Λ of a homeomorphism f admits a Lyapunov function, that is, a continuous map $V\colon U \subset X \to \mathbb{R}$ defined on a neighborhood of Λ such that:*

1. *$V(x) = 0$ if $x \in \Lambda$,*
2. *$V(x) > V(f(x))$ if $x, f(x) \in U \setminus \Lambda$.*

Proof. Consider $\phi\colon \mathbb{R}\times X_f \to X_f$ the suspension of f. Consider $i\colon X \to X_f$ a homeomorphism onto its image such that $i(X)$ is a global cross section of ϕ. It is easy to see that Λ is an isolated set for f if and only $\Lambda_f = \phi_{\mathbb{R}}(i(\Lambda))$ is an isolated set for ϕ. Now consider a Lyapunov function V' for Λ_f. A Lyapunov function for f can be defined by $V(x) = V'(i(x))$. □

Recall that $\mathbb{K}$ denotes the compact metric space of compact subsets of X with the Hausdorff metric. Denote by $\mathcal{F}_1 = \{A \in \mathbb{K} : |A| = 1\}$ where $|A|$ denotes the cardinality of A. Given a homeomorphism $f\colon X \to X$ define the homeomorphism $f'\colon \mathbb{K} \to \mathbb{K}$ as $f'(A) = \{f(x) : x \in A\}$. Notice that $\mathcal{F}_1$ is invariant under f'.

Corollary 3.6.12. *For a homeomorphism $f\colon X \to X$ the following statements are equivalent:*

1. *f is an expansive homeomorphism,*
2. *$\mathcal{F}_1$ is an isolated set for f',*
3. *there is a continuous function $V\colon U \subset \mathbb{K} \to \mathbb{R}$ defined on a neighborhood of $\mathcal{F}_1$ such that $V(A) = 0$ if and only if $A \in \mathcal{F}_1$ and $V(A) > V(f'(A))$ if $A, f'(A) \in U \setminus \mathcal{F}_1$.*

Proof. $(1 \to 2)$. Let δ be an expansive constant and define

$$U = \{A \in \mathbb{K} : \operatorname{diam}(A) < \delta\}.$$

It is easy to see that U is an isolating neighborhood of $\mathcal{F}_1$.

$(2 \to 3)$. It follows by Theorem 3.6.11.

$(3 \to 1)$. Take $\delta > 0$ such that if $\operatorname{dist}(x, y) \leq \delta$ then $\{x, y\} \in U$. Let us prove that δ is an expansive constant for f. Assume by contradiction that $\operatorname{dist}(f^n(x), f^n(y)) \leq \delta$ for all $n \in \mathbb{Z}$ and $x \neq y$. Define $A = \{x, y\}$. We have that $V(f'^n(A))$ is a decreasing sequence. Without loss of generality assume that $V(A) < 0$. Suppose that $f'^n(A)$ accumulates in B. Now it is easy to see that $B \in U \setminus \mathcal{F}_1$ and also $V(B) = V(f'(B))$. This contradiction proves the theorem. □

Recall that a *continuum* is a compact connected set. Denote by $\mathcal{C}(X) = \{C \in \mathbb{K} : C \text{ is connected}\}$ the space of continua of X.

Corollary 3.6.13. *For a homeomorphism $f\colon X \to X$ the following statements are equivalent:*

1. *f is a continuum-wise expansive homeomorphism,*
2. *$\mathcal{F}_1$ is an isolated set for $f'\colon \mathcal{C}(X) \to \mathcal{C}(X)$,*
3. *there is a continuous function $V\colon U \subset \mathcal{C}(X) \to \mathbb{R}$ defined on an open set $U \subset \mathcal{C}(X)$ containing $\mathcal{F}_1$ such that $V(A) = 0$ if $A \in \mathcal{F}_1$ and $V(A) > V(f'(A))$ if $A, f'(A) \in U \setminus \mathcal{F}_1$.*

Proof. The proof is similar to the proof of Corollary 3.6.12. □

3.7 Examples

We have developed several techniques to prove expansiveness. They will be applied in the following examples.

3.7.1 The shift map

The shift map is an abstract dynamical system. It is defined on a symbolic space. Define $I_j = \{0, 1, 2, \dots, j-1\}$ and consider

$$\Sigma_j = \{a\colon \mathbb{Z} \to I_j\}.$$

An element of Σ_j is a sequence of elements in I_j. The value $a(n)$ for $n \in \mathbb{Z}$ is denoted as a_n. Consider the function $N\colon \Sigma_j \to \mathbb{Z} \cup \{\infty\}$ defined as

$$N(a,b) = \begin{cases} \infty & \text{if } a = b, \\ \min\{|n| : a_n \neq b_n, n \in \mathbb{Z}\} & \text{if } a \neq b. \end{cases} \tag{3.13}$$

Note the relationship between this function and the one defined in Eq. (3.1) (related with uniform expansiveness and the construction of a hyperbolic metric). In Σ_j we consider the metric

$$\operatorname{dist}(a,b) = \lambda^{-N(a,b)}$$

for $\lambda > 1$. It is known that $(\Sigma_j, \operatorname{dist})$ is a compact metric space homeomorphic to the Cantor set. The *shift* map is $\sigma\colon \Sigma_j \to \Sigma_j$ defined as $(\sigma(a))_n = a_{n-1}$. It is easy to see that f is a homeomorphisms. Moreover, it is an expansive one, with expansive constant $\delta = 1/2$. In fact λ is an expanding factor of the metric, so dist is a hyperbolic metric. A *subshift* is the restriction of the shift to a compact invariant subset of Σ_j. Of course, subshifts are expansive too. As we will see in Theorem 4.1.12 (a result from [65]) every expansive homeomorphism can be covered with a subshift.

The hyperbolicity of the subshifts was used by Walters in [134] to obtain the structural stability (in a special sense) of some subshifts. See the reference for more details.

Sturmian subshifts

Let $\beta \in (0,1)$ be an irrational number. In Σ_2 consider the sequence a defined as

$$a_n = \begin{cases} 0 & \text{if } n\beta - k \in [0,\beta) \text{ for some } k \in \mathbb{Z}, \\ 1 & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{Z}$. Consider Ω as the closure of the orbit of a by the shift map. By definition $f = \sigma|\Omega$ is a subshift and it is called as a *Sturmian subshift*. It is an exercise to check that it is minimal. Note that $a_0 = 0$ and $a_1 = 1$. Consider $n_j, k_j \in \mathbb{Z}$ satisfying $n_j\beta - k_j \to 1^-$. Let $b = \lim_{j\to\infty} f^{n_j}(a)$. Note that $b_0 = 1$, $b_1 = 1$ and $b_n = a_n$ if $n \notin \{0,1\}$. Therefore we have that $\operatorname{dist}(f^n(a), f^n(b)) \to 0$ as $n \to \pm\infty$. This is the doubly asymptotic pair found by Hedlund in [49]. It can also be proved that if $\operatorname{dist}(f^n(c), f^n(d)) \to 0$ as $n \to +\infty$ or $n \to -\infty$, with $c \neq d$, then $\{c,d\} = \{f^n(a), f^n(b)\}$ for some $n \in \mathbb{Z}$. This implies that Sturmian subshifts are positive and negative 2-expansive.

These subshifts are conjugate with the non-wandering set of Denjoy's counterexamples.

Interval exchange subshifts

Consider $0 < \alpha < \beta < 1$ and define $r\colon I = [0,1) \to I$ as

$$r(x) = \begin{cases} x + 1 - \alpha & \text{if } x \in [0,\alpha), \\ x - \alpha + 1 - \beta & \text{if } x \in [\alpha,\beta), \\ x - \beta & \text{if } x \in [\beta,1). \end{cases}$$

It is an *interval exchange map* and is illustrated in Figure 3.1. It is known that if α, β are

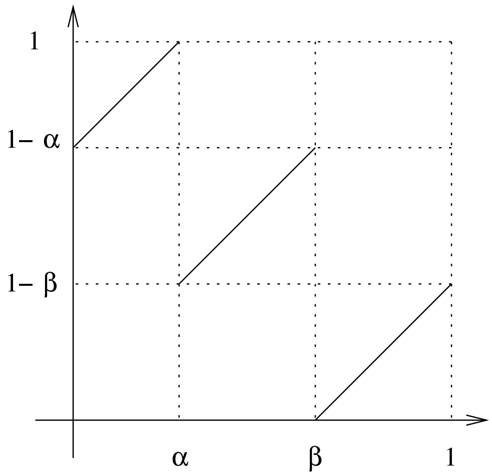

Figure 3.1: An interval exchange map.

rationally independent then r is minimal. See, for example [129] for a proof and more on the minimal interval exchange maps. Define the sequence on three symbols $a \in \Sigma_3$ as

$$a_n = \begin{cases} 0 & \text{if } r^n(0) \in [0,\alpha), \\ 1 & \text{if } r^n(0) \in [\alpha,\beta), \\ 2 & \text{if } r^n(0) \in [\beta,1). \end{cases}$$

As for Sturmian subshifts, define Ω as the closure of the orbit of a by the shift map of Σ_3 and consider $f = \sigma|\Omega$. The properties of f that we wish to remark are: 1) there are no doubly asymptotic pairs of points and 2) there are no $x, y, z \in \Omega$ such that x, y are positive asymptotic and y, z are negative asymptotic. Of course, f is expansive.

3.7.2 Quasi-hyperbolic sets

The following exposition follows [75]. Let $f\colon M \to M$ be a C^1 diffeomorphism of a compact smooth Riemannian manifold M. Suppose that $K \subset M$ is a compact f-invariant set, i.e. $f(K) = K$.

Definition 3.7.1. We say that K is a *quasi-hyperbolic set* if for all $v \in T_K M$ the set $\{\|df^n v\| : n \in \mathbb{Z}\}$ is bounded only if $v = 0$.

For example Anosov diffeomorphisms, quasi-Anosov diffeomorphisms and hyperbolic sets are quasi-hyperbolic. We will show that quasi-hyperbolicity implies expansiveness. Assume that $K \subset M$ is a quasi-hyperbolic set for f.

Lemma 3.7.2 (Lewowicz [72]). *There is $m \geq 0$ such that $\max\{\|df^m u\|, \|df^{-m}u\|\} > 2\|u\|$ for all $u \neq 0$, $u \in T_K M$.*

Proof. Since $S_K M = \{v \in T_K M : \|v\| = 1\}$ is compact there is $N > 0$ such that for all $u \in S_K M$, there is n such that $\|df^n u\| > 2$ and $|n| \leq N$. Therefore, for all $u \in T_K M$, $u \neq 0$, it holds that $\|df^n u\| > 2\|u\|$ for some n with $|n| \leq N$. Let $l > 0$ such that $\|df^n u\| \geq l\|u\|$ for all $u \in T_K M$ with $u \neq 0$ and $|n| \leq N$. Take $j > 0$ such that $2^j l > 2$. We will show that $m = jN$ satisfies the thesis of the Lemma.

For $u_o \in T_K M$, $u_0 \neq 0$, take n_0 such that $|n_0| \leq N$ and $\|df^{n_0} u_0\| \geq \|df^k u_0\|$ if $|k| \leq N$. It implies that $\|df^{n_0} u_0\| > 2\|u_0\|$. We will show that if n_0 is positive then $\|df^m u_0\| > 2\|u_0\|$. In the same way it could be proved that if n_0 is negative then $\|df^{-m} u_0\| > 2\|u_0\|$. So, suppose that $n_0 > 0$. Let $u_1 = df^{n_0} u_0$ and applying the previous remarks we find n_1 such that $|n_1| \leq N$, $\|df^{n_1} u_1\| \geq \|df^k u_1\|$ if $|k| \leq N$ and $\|df^{n_1} u_1\| > 2\|u_1\|$. Notice that n_1 must be positive. Therefore $\|f^{n_0+n_1} u_0\| > 2^2\|u_0\|$. By induction we find a sequence n_ν, $\nu = 0, 1, 2, \dots$ such that $0 < n_\nu \leq N$ and

$$\|df^{\sum_{\nu=0}^{k-1} n_\nu} u_0\| > 2^k\|u_0\|.$$

Take $k \geq j$ such that $\sum_{\nu=0}^{k-1} n_\nu \leq m < \sum_{\nu=0}^{k} n_\nu$. Then

$$\|df^m u_0\| \geq 2^k l\|u_0\| \geq 2^j l\|u_0\| \geq 2\|u_0\|$$

as we wanted to prove. □

Now consider the quadratic form $B\colon T_K M \to \mathbb{R}$ defined as

$$B(u) = \sum_{i=0}^{m-1} \|df^{m+i} u\|^2 - \sum_{i=0}^{m-1} \|df^i u\|^2$$

Proposition 3.7.3. *For all $u \in T_K M$, $u \neq 0$, it holds that*

$$B(dfu) - B(u) > 0.$$

Proof. Note that

$$B(dfu) - B(u) = \|df^{2m} u\|^2 - 2\|df^m u\| + \|u\|^2.$$

Therefore, the proposition follows by Lemma 3.7.2. □

Now take $r > 0$ such that for all $p \in K$ the exponential map $\exp_p$ restricted to the r ball in T_pM is a diffeomorphism onto its image. Take $\delta_0 > 0$ such that $B_{\delta_0}(p) \subset \exp_p(T_p^r)$ for all $p \in K$, where

$$T_p^r = \{u \in T_pM : \|u\| \leq r\}.$$

Let $N_\delta = \{(p, q) \in K \times K : \text{dist}(p, q) \leq \delta\}$ and define $V\colon N_\delta \to \mathbb{R}$, for $\delta \in (0, \delta_0)$, as

$$V(x, y) = B(\exp_x^{-1}(y)).$$

Proposition 3.7.4. *If δ is small enough then V is a Lyapunov function on N_δ. Therefore, quasi-hyperbolic sets are expansive.*

Proof. Define $B'\colon T_KM \to \mathbb{R}$ as $B'(u) = B(dfu) - B(u)$. It is a positive quadratic form by Proposition 3.7.3. Therefore, there is $\gamma > 0$ such that for all $u \neq 0$ it holds that

$$\frac{B'(u)}{\|u\|^2} > \gamma.$$

Consider $\tilde{f}_x = \exp_{fx}^{-1} \circ f \circ \exp_x$ defined for small vectors in the tangent space at x. We have that $d_xf = d_0\tilde{f}$ if we identify T_xM with T_0T_xM. Therefore, if $r(u) = dfu - \tilde{f}u$ then

$$\frac{r(u)}{\|u\|} \to 0$$

as $u \to 0$. We want to prove that $B(\tilde{f}u) - B(u) > 0$. So

$$\frac{B(\tilde{f}u) - B(u)}{\|u\|^2} = \frac{B(\tilde{f}u) - B(dfu)}{\|u\|^2} + \frac{B'(u)}{\|u\|^2}.$$

Recall that $B'(u)/\|u\|^2 > \gamma > 0$ for all $u \neq 0$, therefore it is enough to show that

$$\frac{B(\tilde{f}u) - B(dfu)}{\|u\|^2} \to 0 \tag{3.14}$$

as $u \to 0$. Let w be a symmetric bilinear form such that $B(v) = w(v, v)$. Then

$$\begin{aligned}\frac{B(\tilde{f}u) - B(dfu)}{\|u\|^2} &= \frac{w(dfu + r(u), dfu + r(u)) - w(dfu, dfu)}{\|u\|^2} \\ &= 2w\left(\frac{dfu}{\|u\|}, \frac{r(u)}{\|u\|}\right) - w\left(\frac{r(u)}{\|u\|}, \frac{r(u)}{\|u\|}\right)\end{aligned}$$

Since $dfu/\|u\|$ is bounded and $r(u)/\|u\| \to 0$ we have proved (3.14). So, δ must be chosen in such a way that if $\text{dist}(x, y) < \delta$ then

$$\frac{B(\tilde{f}u) - B(dfu)}{\|u\|^2} < \gamma$$

if $u = \exp_x^{-1} y$. We have proved that V is a Lyapunov function. Now applying Theorem 3.6.4 we conclude that quasi-hyperbolic sets are expansive. □

A diffeomorphism is *quasi-Anosov* if M is a quasi-hyperbolic set. Mañé in [77] proved that a diffeomorphism is robustly expansive in the C^1 topology if and only if it is quasi-Anosov. A diffeomorphism is *robustly expansive* if it is in the interior of the set of expansive C^1 diffeomorphisms with the C^1 topology.

3.7.3 Pseudo-Anosov diffeomorphisms

Let S be a compact topological surface without boundary. A *flat structure* is a finite family of homeomorphisms onto its images $\phi_\alpha\colon D = [0,1]\times[0,1] \to S$, $\alpha \in A$, such that $S = \cup_{\alpha\in A}\phi(D)$, for all $x \in S$, and the change of coordinates are local isometries preserving vertical and horizontal lines in D.

The vertical and the horizontal foliations of D induce two foliations on S that may have singular points. Also a flat Riemannian metric is induced away from the singularities. A *pseudo-Anosov* homeomorphism of S is a homeomorphism $f\colon S \to S$ that preserves both foliations and

- $\mathrm{Len}(f^{-1}(\gamma_s)) = \lambda\,\mathrm{Len}(\gamma_s)$ for every vertical arc of leaf γ_s and
- $\mathrm{Len}(f(\gamma_u)) = \lambda\,\mathrm{Len}(\gamma_u)$ for every horizontal arc of leaf γ_u,

where $\lambda > 1$ is a fixed parameter.

Proposition 3.7.5. *If f is pseudo-Anosov then the distance induced by the flat Riemannian metric is hyperbolic. Consequently, pseudo-Anosov homeomorphisms are expansive.*

Proof. It follows by the hyperbolic behavior of vertical and horizontal leaves. □

Let us give another related example that is not expansive but it is cw-expansive. Consider the *torus* $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $f\colon T^2 \to T^2$ be defined as $f(x,y) = (2x+y, x+y)$ for all $(x,y) \in T^2$. It is a linear Anosov diffeomorphism. If we identify $p \simeq -p$ we obtain a sphere $S = T^2/\simeq$. From the definitions it is easy to check that f induces a map g on S as $g([p]) = [f(p)]$, where $[\,]$ denotes the equivalence class. We have that g is not expansive, in fact we will see that there are no expansive homeomorphisms on the sphere. It is interesting because it is cw-expansive.

Chapter 4

Variations of expansiveness

4.1 Stability and Dimension

4.1.1 Positive continuum-wise expansiveness

Definition 4.1.1. We say that f is *positive continuum-wise expansive* if there is $\delta > 0$ such that if $\operatorname{diam}(f^n(C)) \leq \delta$ for all $n \geq 0$ for some continuum C then C is a singleton. In this case we say that δ is a *positive cw-expansive constant.*

The solenoid is an example of a positive continuum-wise expansive homeomorphism. We will show that if f is positive cw-expansive and X is locally connected then X is a finite set. In spite that this result is not stated in [75], our exposition follows the ideas therein.

Theorem 4.1.2. *If f is positive cw-expansive and X is locally connected then X is a finite set.*

Proof. Note that positive cw-expansiveness implies that the space of singletons is a repeller isolated set for the action of f on continua subsets of X. Therefore, by Propositions 3.1.2 and 3.1.5, there is δ_0 such that if $\operatorname{diam}(C) < \delta_0$ for a continuum $C \subset X$, then $\operatorname{diam}(f^n(C)) \to 0$ as $n \to -\infty$. For each $x \in X$ take a continuum C_x such that x is in the interior of C_x and $\operatorname{diam}(C_x) \leq \delta_0$. Since X is compact and the interiors of C_x cover X there are different points $x_1, \dots x_n \in X$ such that $X = \cup_{i=1}^n C_n$ where $C_n = C_{x_n}$. We known that $\operatorname{diam}(f^{-j}(C_i)) \to 0$ as $j \to \infty$. Also, since f is a homeomorphism, we know that $f^{-j}(C_1), \dots, f^{-j}(C_n)$ cover X. This easily implies that X has at most n points. □

4.1.2 Lyapunov stability

In this section assume that X is a *Peano continuum* (compact, connected and locally connected metric space). Also, consider $f\colon X \to X$ a continuum-wise expansive homeomorphism. We will show that with this hypothesis there are no Lyapunov stable points.

Proposition 4.1.3 (Kato [61]). *If f is cw-expansive then for all $\varepsilon > 0$ there is $\delta > 0$ such that if $C \subset X$ is a continuum,* $\operatorname{diam}(C) \leq \delta$ *and* $\operatorname{diam}(f^n C) \geq \varepsilon$ *for some $n > 0$ then* $\operatorname{diam}(f^m(C)) \geq \delta$ *for all $m > n$.*

Proof. We will argue by contradiction. Then there is a cw-expansive constant $\varepsilon > 0$, a positive sequence $\delta_k \to 0$, a sequence of continua $C_k \subset X$ and integer sequences $m_k > n_k > 0$ such that $\operatorname{diam}(C_k) \leq \delta_k$, $\operatorname{diam}(f^{n_k}(C_k)) \geq \varepsilon$, $\operatorname{diam}(f^{m_k}(C_k)) \leq \delta_k$. In this case we have that $n_k \to +\infty$ and $m_k - n_k \to +\infty$. Eventually taking a subcontinuum of C_k we can also assume that $\operatorname{diam}(f^j(C_k)) \leq \varepsilon$ for all $j = 0, 1, 2, \dots, m_k$. Now, a limit continuum, in the Hausdorff metric, of the sequence $f^{n_k}(C_k)$ contradicts that ε is a cw-expansive constant. □

Remark 4.1.4. *This property of cw-expansive homeomorphisms suggests to define: f is* super expansive *if there is $\delta > 0$ such that if* $\operatorname{dist}(x, y) \in (0, \delta)$ *then there is $n \in \mathbb{Z}$ such that if $n > 0$ then for all $m \geq n$ it holds that* $\operatorname{dist}(f^m(x), f^m(y)) > \delta$ *and if $n < 0$ then there is $m \leq n$ such that* $\operatorname{dist}(f^m(x), f^m(y)) > \delta$. *Super expansiveness implies expansiveness and also that there are no doubly asymptotic points. What else can be said about such homeomorphisms?*

Remark 4.1.5. *A well known result in continuum theory states that every Peano continuum admits a metric, called convex metric, for which every ball is connected. See* [92] *for more on this topic.*

Theorem 4.1.6 (Lewowicz, Hiraide, Kato). *Non-trivial Peano spaces admit no cw-expansive homeomorphisms with stable points.*

Proof. Arguing by contradiction assume that $x \in X$ is a stable point of the cw-expansive homeomorphism $f : X \to X$ of the non-trivial Peano continuum X. By Remark 4.1.5 we will assume that every ball is connected. Let $\alpha > 0$ be a cw-expansive constant of f. Take $\varepsilon \in (0, \alpha)$. Since x is a stable point we know that there is $\delta > 0$ such that if $\operatorname{dist}(y, x) < \delta$ then $\operatorname{dist}(f^n(y), f^n(x)) < \varepsilon$ for all $n \geq 0$.

In this paragraph we will show that there is $\sigma \in (0, \delta)$ such that

$$f^n(B_\sigma(f^{-n}(x))) \subset B_\delta(x), \forall n \geq 1. \tag{4.1}$$

By contradiction assume that this is not the case. Then, for $\sigma = 1/k$, $k \geq 1$, there is n_k such that $f^{n_k}(B_{1/k}(f^{-n_k}(x)))$ is not contained in $B_\delta(x)$. In this case, it is easy to see that $n_k \to +\infty$ as $k \to +\infty$. Also, we can take a continuum $C_k \subset B_{1/k}(f^{-n_k}(x))$ such that $f^{-n_k}(x) \in C_k$, $\operatorname{diam}(f^i(C_k)) \leq \delta$ for all $i = 1, \dots, n_k$ and for some $i_k \in \{1, \dots, n_k\}$ the equality $\operatorname{diam}(f^{i_k}(C_k)) = \delta$ holds. Since x is stable, we know that $\operatorname{diam}(f^j(B_\delta(x)) \leq \varepsilon < \alpha$ for all $j \geq 0$, where α is a cw-expansive constant. If we define $D_k = f^{i_k}(C_k)$ we have that D_k is a continuum, $\operatorname{diam}(D_k) = \delta$ and $\operatorname{diam}(f^j(D_k)) \leq \varepsilon$ for all $j \geq -n_k$. Then, taking a limit continuum of D_k, in the Hausdorff metric, we contradict that f is cw-expansive with expansive constant α.

We have that (4.1) implies that every point in the set $\alpha(x)$ is stable. Let $z \in \alpha(x)$. Since z is stable, it is easy to see that $x \in \omega(z)$. We also have by the stability of z and the fact that $z \in \alpha(x)$ that $\omega(x) = \omega(z)$. Then $x \in \omega(x)$, i.e., x is a recurrent point. Since x is stable, X is locally connected and f is cw-expansive we have that x is in fact asymptotically stable. Therefore, there are $r > 0$ small and $m \geq 0$ such that $f^m(B_r(x)) \subset B_r(x)$. Since $\operatorname{diam}(f^i(B_r(x))) \to 0$ as $i \to +\infty$, we have that $\cap_{i\geq 0} f^{im}(B_r(x))$ is a singleton $\{y\}$. In particular $f^m(y) = y$, y is stable and $y \in \omega(x)$. If x is not in the orbit of y we have a contradiction because x is recurrent and $y \in \omega(x)$ with y a stable point. This proves that x is a periodic point.

Therefore, we have proved that every stable point is periodic. This is a contradiction because the set of stable points is an open set. This contradiction proves the Theorem. □

4.1.3 Stable sets

In this section we study basic topological properties of stable and unstable sets of cw-expansive homeomorphisms. Let $f\colon X \to X$ be a cw-expansive homeomorphisms.

Definition 4.1.7. A continuum C is *stable* if $\operatorname{diam}(f^k(C)) \leq \varepsilon$ for all $k \geq 0$ with ε a cw-expansive constant. If f is cw-expansive and C is a stable continuum then $\operatorname{diam}(f^k(C)) \to 0$.

Proposition 4.1.8. *If $f\colon X \to X$ is cw-expansive and X is a Peano continuum then there is $\delta > 0$ such that for all $x \in X$ there is a stable continuum C containing x with diameter δ.*

Proof. Given a cw-expansive constant ε consider δ from Proposition 4.1.3. For $x \in X$ consider $z \in \omega(x)$ and take a compact connected neighborhood U of z such that $\operatorname{diam}(U) \leq \delta$. Take $n_k \to \infty$ such that $f^{n_k}x \in U$. By Theorem 4.1.6 we have that z is not stable for f^{-1}. So there is $N > 0$ such that $\operatorname{diam}(f^{-N}(U)) > \varepsilon$. Suppose that $n_k > N$ for all k. Consider $C_k \subset U$ such that C_k is a continuum, $f^{n_k}(x) \in C_k$, $\operatorname{diam}(f^{-j}(C_k) \leq \varepsilon$ for all $j = 0, 1, \dots, n_k$ and for some of this values of j, say j_k, the equality holds. By Proposition 4.1.3 we know that $\operatorname{diam}(f^{-n}(C_k) \geq \delta$ for all $n \geq j_k$. Note that $x \in f^{-n_k}(C_k)$. Then a limit continuum of $f^{-n_k}(C_k)$ satisfies the thesis. □

I learned the following argument from Lewowicz.

Corollary 4.1.9 (Kawamura [64])**.** *If X is a Peano continuum admitting a cw-expansive homeomorphism then no open subset of X is homeomorphic with $\mathbb{R}$. In particular, the circle, the interval and surfaces with non-empty boundary do not admit cw-expansive homeomorphisms.*

Proof. Suppose by contradiction that there is an open subset U homeomorphic to $\mathbb{R}$. By Proposition 4.1.8 for $x \in U$ there is a stable continuum C of positive diameter. Since U is homeomorphic to $\mathbb{R}$ we have that C has non-empty interior. The points in the interior of C are stable points, but this contradicts Theorem 4.1.6.

The cases of the circle and the interval follows easily. If X is a surface with boundary, the boundary is a f-invariant union of circles. And f should be expansive restricted to the boundary, a contradiction. □

We say that $C \subset X$ *separates* if $X \setminus C$ is not connected. The following proof works on every Peano continuum with the following property: every point has arbitrary small neighborhoods with connected boundary. For example if X is a manifold of dimension greater than 1. According to the previous result, one dimensional manifolds need not to be consider. The next result is very important in the study of surface homeomorphisms.

Proposition 4.1.10. *If $f\colon X \to X$ is cw-expansive and X is a connected manifold then no stable set separates X.*

Proof. Consider a cw-expansive constant $\delta \in (0, \operatorname{diam}(X)/2)$. By contradiction suppose there is a stable set C such that $Y = X \setminus C$ is not connected. Let $\mathcal{U}$ be a finite cover of X such that $\operatorname{diam} U < \delta$ and ∂U is connected for all $U \in \mathcal{U}$. Since $\delta < \operatorname{diam} X/2$ we have that $\operatorname{diam}(X \setminus U) > \delta$ for all $U \in \mathcal{U}$. If $\gamma \in (0, \delta)$ is a Lebesgue number for $\mathcal{U}$ we can suppose that $\operatorname{diam} f^n(C) < \gamma$ for all $n \geq 0$ (eventually changing C for f^nC for some $n \geq 0$ if needed). So, for all $n \geq 0$ there is $U \in \mathcal{U}$ such that $f^n(C) \subset U$. Since X is locally connected and Y is an open set we have that the connected components of Y are open sets. For each $n \geq 0$ consider $U_n \in \mathcal{U}$ such that $f^nC \subset U_n$. Since ∂U_n is connected and disjoint with f^nC we have that ∂U_n is contained in one connected component of f^nY. Let Y_n be the connected component of f^nY that contains ∂U_n. Since $\delta < \operatorname{diam} X/2$ and $\operatorname{diam} U_n < \delta$ we have that $\operatorname{diam} X \setminus U_n > \delta$, so, $\operatorname{diam} Y_n > \delta$ for all $n \geq 0$.

We will show that there is n_0 such that if $m \geq n \geq n_0$ we have that $Y_m = f^{m-n}Y_n$. If this were not the case there is some Y_k such that for some $n_1 < 0 < n_2$ we have that $f^{n_1}Y_k \subset U_{k-n_1}$ and $f^{n_2}Y_k \subset U_{k+n_2}$. So, $f^{n_1}Y_k$ contradicts Proposition 4.1.3.

Therefore there is a connected component of Y, say Y', such that for all $n \geq n_0$ it holds that $f^nY' \subset U_n$. Then $\operatorname{diam} f^n(Y') < \delta$ for all $n \geq n_0$ and the points of Y' are stable. This contradicts Proposition 4.1.6. □

4.1.4 Coverings

A *covering* is a finite family of open sets $\mathcal{U} = \{U_i \subset X : i = 1, \dots n\}$ such that $\cup_{i=1}^n U_i = X$. Define

$$\|\mathcal{U}\| = \sup_{U \in \mathcal{U}} \operatorname{diam} U.$$

A sequence of coverings $\mathcal{U}_n$ is *cofinal* if $\|\mathcal{U}_n\| \to 0$ as $n \to \infty$. A covering $\mathcal{U}$ is a *generator* for f if for every bisequence $\{A_n\}_{n\in\mathbb{Z}}$ of members of $\mathcal{U}$ we have that

$$\cap_{n\in\mathbb{Z}} f^n(\operatorname{clos}(A_n))$$

is at most one point. A *weak generator* is defined in the same way, but requiring that

$$\cap_{n\in\mathbb{Z}} f^n(A_n)$$

is at most one point.

The definition of generators were introduced by Keynes and Robertson [65], we are following the exposition of [3] page 37.

Theorem 4.1.11. *The following statements are equivalent:*

1. *f is expansive,*
2. *f has a generator and*
3. *f has a weak generator.*

Proof. ($1 \to 2$) Let $\delta > 0$ be an expansive constant for f and let $\mathcal{U}$ be a finite covering such that $\|\mathcal{U}\| < \delta$. Suppose that $x, y \in \cap_{n\in\mathbb{Z}} f^n(\mathrm{clos}(A_n))$ for some bisequence $A_n \in \mathcal{U}$. Then $\mathrm{dist}(f^n x, f^n y) \leq \delta$ for all $n \in \mathbb{Z}$ and expansiveness implies that $x = y$.

($2 \to 3$) Is trivial.

($3 \to 1$) Suppose that $\mathcal{U}$ is a weak generator. Let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$. If $\mathrm{dist}(f^n x, f^n y) < \delta$ for all $n \in \mathbb{Z}$ then for each $n \in \mathbb{Z}$ there is $A_n \in \mathcal{U}$ such that $f^{-n}x, f^{-n}y \in A_n$. So, $x, y \in \cap_{n\in\mathbb{Z}} f^n(A_n)$ which is at most one point. Therefore $x = y$ and f is expansive. □

Theorem 4.1.12 (Keynes and Robertson [65])**.** *If $f\colon X \to X$ is an expansive homeomorphism then there is a subshift $\sigma'\colon Y \to Y$ and a continuous surjective map $\pi\colon Y \to X$ such that $\pi \circ \sigma = f \circ \pi$.*

Proof. Consider a generator $\mathcal{U} = \{A_0, \dots, A_{d-1}\}$. Let $\sigma\colon \Sigma \to \Sigma$ be the shift on d symbols and define

$$Y = \{m \in \Sigma : \cap_{i\in\mathbb{Z}} f^{-i}(\mathrm{clos}(A_{m_i})) \neq \emptyset\}.$$

We will show that Y is closed. Let m^j be a sequence in Y such that $m^j \to m \in \Sigma$. Denote by y_j the only point in $\cap_{i\in\mathbb{Z}} f^{-i}(\mathrm{clos}\, A_{m_i^j})$. Eventually taking a subsequence we can assume that $y_j \to y \in X$. Let $j_i \geq 0$ be such that $m_i^j = m_i$ for $j \geq j_i$. Thus $y_j \in f^{-i}(\mathrm{clos}(A_{m_i})$ for all $j \geq j_i$ and $y \in f^{-i}(\mathrm{clos}(A_{m_i}))$. Hence $\cap_{i\in\mathbb{Z}} f^{-i}(\mathrm{clos}(A_{m_i})) \neq \emptyset$ and $m \in Y$.

Clearly Y is invariant by the shift σ, since if $\cap_{i\in\mathbb{Z}} f^{-i}(\mathrm{clos}(A_{m_i})) \neq \emptyset$, then

$$\cap_{i\in\mathbb{Z}} f^{-i+1}(\mathrm{clos}(A_{m_i})) = \cap_{i\in\mathbb{Z}} f^{-i}(\mathrm{clos}(A_{m_{i+1}})) = \cap_{i\in\mathbb{Z}} f^{-i}(\mathrm{clos}(A_{\sigma(m)_i})),$$

and $\sigma(m) \in Y$. Define $\pi\colon Y \to X$ by

$$\pi(m) = \cap_{i\in\mathbb{Z}} f^{-i+1}(\mathrm{clos}(A_{m_i})).$$

Then

$$\begin{aligned}\pi\sigma(m) &= \cap_{i\in\mathbb{Z}} f^{-i}(\operatorname{clos}(A_{\sigma(m)_i})) = \cap_{i\in\mathbb{Z}} f^{-i}(\operatorname{clos}(A_{m_{i+1}}))\\ &= f(\cap_{i\in\mathbb{Z}} f^{-i}(\operatorname{clos}(A_{m_i}))) = f\pi(m).\end{aligned}$$

for all $m \in Y$. Since $\mathcal{U}$ is a cover, π is clearly onto. The continuity of π follows by our previous arguments. □

4.1.5 Finite dimension

Let $f\colon X \to X$ be a cw-expansive homeomorphism of a compact metric space.

Definition 4.1.13. We say that the *topological dimension* of X is *finite* if there is $n \in \mathbb{N}$ such that for all $r > 0$ there is an open covering $\mathcal{U}$ of X such that:

- $\operatorname{diam}(U) < r$ for all $U \in \mathcal{U}$ and
- every point of X belongs to at most $n+1$ open sets of $\mathcal{U}$.

In this case we say that $\dim_{top}(X) \leq n$.

In this section we will show that the topological dimension of X is finite, if X admits a cw-expansive homeomorphism $f\colon X \to X$. Mañé in [78] gave a proof for expansive homeomorphisms. In [61] Kato proved the result for cw-expansive homeomorphisms. His proof uses some *basic* topological results that we explain in detail here.

Let $c > 0$ be a cw-expansive constant for f. By Proposition 4.1.3 we know that there is $\delta > 0$ such that:

$$\begin{array}{l}\text{if } C \text{ is a continuum, } \operatorname{diam}(C) \leq \delta \text{ and } \operatorname{diam}(f^n(C)) \geq c \text{ for some } n > 0\\ \text{then } \operatorname{diam}(f^m(C)) \geq \delta \text{ for all } m > n.\end{array} \tag{4.2}$$

Take a finite covering $\{U_1, \dots, U_l\}$ of X by open sets such that $\operatorname{diam}(U_i) < \delta$. We will show that $\dim_{top}(X)$ is at most $l^2 - 1$, where l is the number of elements of the covering. For $n \geq 0$ and $i, j = 1, \dots, l$ define the open set

$$U^n_{i,j} = f^n U_i \cap f^{-n} U_j.$$

Proposition 4.1.14. *For all $\sigma > 0$ there is $n_0 > 0$ such that if $n \geq n_0$ and C is a connected component of the closure of some $U^n_{i,j}$ then* $\operatorname{diam}(C) < \sigma$.

Proof. By contradiction assume that there is $\sigma > 0$, $n_k \to \infty$ and C_k a component of $U^{n_k}_{i_k,j_k}$ such that $\operatorname{diam}(C_k) \geq \sigma$. Let $N > 0$ be such that if $\operatorname{diam}(C) \geq \sigma$ for some continuum $C \subset X$ then $\sup_{|j|\leq N} \operatorname{diam}(f^j(C)) > c$. Since $f^{n_k}(C_k) \subset \operatorname{clos}(U_{j_k})$, $f^{-n_k}(C_k) \subset \operatorname{clos}(U_{i_k})$ and $\operatorname{diam}(U_{i_k}), \operatorname{diam}(U_{j_k}) < \delta$ we have a contradiction with condition (4.2) taking $n_k > N$. □

The following lemmas are topological, no dynamic is involved.

Lemma 4.1.15. *If $Y \subset X$ is a compact subset, $C \subset Y$ is a connected component of Y and U is an open subset containing C then there is an open set V such that $C \subset V \subset U$ and $Y \cap \partial V = \emptyset$.*

Proof. Given a component $C \subset Y$ consider an open set U containing C. For $\varepsilon > 0$ consider A_ε the set of points $a \in Y$ for which there is a finite sequence $x_1, \dots, x_n \in Y$ such that $x_1 \in C$, $x_n = a$ and $\text{dist}(x_i, x_{i+1}) \leq \varepsilon$. Let us show that if ε is sufficiently small then $A_\varepsilon \subset U$. If this is not true, then for all $\varepsilon = 1/k$ there is a chain $x_1^k, x_2^k, \dots, x_{n_k}^k$ such that $x_1^k \in C$, $x_{n_k}^k \notin U$ and $\text{dist}(x_i^k, x_{i+1}^k) < 1/k$. Let $L_k = \{x_1^k, x_2^k, \dots, x_{n_k}^k\}$ and assume that $L_k \to L$ in the Hausdorff metric. It is easy to see that L is a continuum contained in Y that intersects C and the complement of U. Therefore, $L \subset C$, and we have a contradiction with $C \subset U$. Then there is ε such that $A_\varepsilon \subset U$. Finally, we can define $V = U \cap B_\varepsilon(A_\varepsilon)$ and it is easy to see that V satisfies the thesis of the proposition. □

An open covering is a *disjoint covering* if every pair of its members are disjoint.

Lemma 4.1.16. *If $Y \subset X$ is compact and every component of Y has diameter smaller than $\sigma > 0$ then there is a disjoint covering of Y whose members have diameter smaller than σ.*

Proof. Given $x \in Y$ consider C_x the component of x in Y. Let $r > 0$ be so that $B_r(C_x)$ has diameter smaller than σ. From the previous lemma consider an open set V_x such that $C_x \subset V_x \subset B_r(C_x)$ and $\partial V_x \cap Y = \emptyset$. Since Y is compact there are $x_1, \dots, x_n$ such that the corresponding $V_i = V_{x_i}$ form an open covering of Y. Since $\partial V_i \cap Y = \emptyset$ we have that $Y_i = Y \cap V_i$ is a compact set. Define $U_i = \{y \in V_i : \text{dist}(y, Y_i) < (\text{dist}(y, Y_j) \text{ if } i \neq j\}$. In this way $U_i \cap U_j = \emptyset$ if $i \neq j$ and the proof ends. □

Theorem 4.1.17 (Mañé [78], Kato [61])**.** *If X admits a cw-expansive homeomorphism then the topological dimension of X is finite.*

Proof. Given $\sigma > 0$ consider n_0 from Proposition 4.1.14. Take $n \geq n_0$. By the previous lemma we have that each $U_{i,j}^n$ admits a disjoint covering $U_{i,j}^n = \cup_{k=1}^{k=K(i,j,n)} U_{i,j}^{n,k}$ with $\text{diam}(U_{i,j}^{n,k}) < \sigma$. Let

$$\mathcal{U} = \{U_{i,j}^{n,k} : i, j = 1, \dots, l; k = 1, \dots, K(i,j,n)\}.$$

We will show that every point of X belongs to at most l^2 sets of the covering $\mathcal{U}$. Suppose that

$$\bigcap_{m=1,\dots,s} U_{i_m \cdot j_m}^{n,k_m} \neq \emptyset.$$

By construction, if $(i_{m_1}, j_{m_1}) = (i_{m_2}, j_{m_2})$ then

$$U_{i_{m_1} \cdot j_{m_1}}^{n,k_{m_1}} = U_{i_{m_2} \cdot j_{m_2}}^{n,k_{m_2}}.$$

Therefore $s \leq l^2$ and the proof ends. □

Corollary 4.1.18. *If X admits an expansive homeomorphism then X is homeomorphic to a compact subset of an Euclidean space $\mathbb{R}^m$ for some $m \geq 1$.*

Proof. It is known that if $\dim_{top}(X) \leq n$ then X is homeomorphic to a compact subset of $\mathbb{R}^{2n+1}$, see Theorem V2 in [54]. □

4.1.6 Minimal sets

A compact metric space X has *local dimension* 0 at a point $x \in X$ if it has arbitrary small neighborhoods with empty boundary. The space X has *dimension* 0 if it has dimension 0 at every $x \in X$. It is easy to see that X has dimension 0 if and only if it is totally disconnected, i.e. every connected set in X is a singleton. The purpose now is to prove that if X admits a minimal cw-expansive homeomorphism then X has dimension 0.

Let $c > 0$ be a cw-expansive constant of f. We say that a continuum C is *r-stable* if for all $n \geq 0$ it holds that $\operatorname{diam}(f^n(C)) \leq r$.

Lemma 4.1.19. *There exists $m \in \mathbb{N}$ such that if $\Lambda \subset X$ is a c-stable continuum with $\operatorname{diam}(\Lambda) = c/3$ then there are c-stable continua $\Lambda_1, \Lambda_2 \subset f^{-m}(\Lambda)$ satisfying:*

1. $\operatorname{diam}(\Lambda_1) = \operatorname{diam}(\Lambda_2) = c/3$,
2. $\inf\{\operatorname{dist}(z, w) : z \in \Lambda_1, w \in \Lambda_2\} \geq c/3$,

Proof. Take m such that if Λ is a c-stable continuum and $\operatorname{diam}(\Lambda) = c/3$ then $\operatorname{diam}(f^{-m}\Lambda) > c$. Such value of m exists by Proposition 3.1.4. Then we can find points $a, b \in f^{-m}(\Lambda)$ such that $\operatorname{dist}(a, b) = c$. Take $\Lambda_1, \Lambda_2 \subset f^{-m}(\Lambda)$ such that $a \in \Lambda_1$, $b \in \Lambda_2$ and $\operatorname{diam}(\Lambda_1) = \operatorname{diam}(\Lambda_2) = c/3$. Let us explain how to take Λ_1. Since $f^{-m}(\Lambda)$ is a continuum, there is a function $g\colon [0, 1] \to \mathbb{K}$ that is continuous with respect to the Hausdorff metric on $\mathbb{K}$, $g(0) = \{a\}$, $g(1) = f^{-m}(\Lambda)$ and $a \in g(t)$ is a subcontinuum of $f^{-m}(\Lambda)$ for all $t \in [0, 1]$. This is a well known result in continuum theory and a proof can be found in [92]. Since $\operatorname{diam}\colon \mathbb{K} \to \mathbb{R}$ is continuous, there is $t_0 \in (0, 1)$ such that $\operatorname{diam}(g(t_0)) = c/3$. Define $\Lambda_1 = g(t_0)$. A similar procedure gives us Λ_2. If there are $z \in \Lambda_1$ and $w \in \Lambda_2$ such that $\operatorname{dist}(z, w) < c/3$ then

$$\operatorname{dist}(a, b) \leq \operatorname{dist}(a, z) + \operatorname{dist}(z, w) + \operatorname{dist}(w, b) < c/3 + c/3 + c/3.$$

This constradicts that $\operatorname{dist}(a, b) = c$ and finishes that proof. □

Remark 4.1.20. *Recall that we have proved in Proposition 3.4.9 that if X is not totally disconnected, i.e.* $\dim(X) > 0$, *then there is a c-stable continuum Λ for f or for f^{-1}. Without loss of generality we will assume that Λ is c-stable for f. Taking a negative iterate of Λ, and eventually a sub-continuum of Λ, we can also assume that* $\operatorname{diam}(\Lambda) = c/3$.

Let U be an open set with $\operatorname{diam}(U) < c/3$. Define $C_0 = \Lambda$ given by Remark 4.1.20. Take Λ_1 and Λ_2 from the previous lemma. Then, either Λ_1 or Λ_2 does not intersect U. Suppose $\Lambda_1 \cap U = \emptyset$. Define $C_1 = \Lambda_1$. Now consider $\Lambda' = C_1$ and apply the lemma, obtaining again Λ'_1 and Λ'_2. Similarly, Λ'_1 or Λ'_2 does not intersect U. Suppose $\Lambda'_2 \cap U = \emptyset$ and define $C_2 = \Lambda'_2$. Using this method we find $C_1, C_2, \ldots$ such that $C_{j+1} \subset f^{-m}(C_j)$ and $C_j \cap U = \emptyset$ for all $j \geq 1$. Take $x \in \cap_{j\geq 0} f^{jm}(C_j)$. Then, $f^{-jm}(x) \notin U$ for all $j \geq 1$.

We have proved:

Proposition 4.1.21. *If f is cw-expansive, Λ is a c-stable continuum,* $\operatorname{diam}(\Lambda) = c/3$ *and U is an open set with* $\operatorname{diam}(U) < c/3$ *(with c a cw-expansive constant) then is $x \in \Lambda$ such that $f^{-jm}(x) \notin U$ for all $j \geq 1$, where m is given by the lemma.*

Theorem 4.1.22 (Mañé [78], Kato [61])**.** *If $f\colon X \to X$ is a minimal cw-expansive homeomorphism then the topological dimension of X is 0.*

Proof. Assume by contradiction that $\dim(X) > 0$. Then, we can apply Remark 4.1.20 to obtain a c-stable continuum Λ of diameter $c/3$. By the previous proposition we know that f^m is not minimal. But, since f is minimal, it is known that X is a disjoint union of $X_1, \ldots, X_m$ compact subsets of X and $f^m\colon X_i \to X_i$ is minimal for all $i = 1, \ldots, m$. Since Λ is connected and the sets X_i form a partition of X of closed sets we have that Λ must be contained in some X_i, say $\Lambda \subset X_1$. If we take U contained in X_1, we arrive to a contradiction with Proposition 4.1.21 because $f^m\colon X_1 \to X_1$ is minimal. □

Corollary 4.1.23. *Every minimal cw-expansive homeomorphism is conjugate with a subshift.*

Proof. It is a consequence of Theorem 4.1.22 and the techniques in the proof of Theorem 4.1.12. □

4.2 Hyper-expansiveness

The purpose of the present section is to study the expansiveness of the induced map f_* on compact subsets. Let $f\colon X \to X$ be a homeomorphism of a compact metric space and consider $\mathbb{K}$ as the hyper-space of X, i.e., the compact metric space with the Hausdorff metric. Define $C(X) = \{Y \in \mathbb{K} : Y \text{ is connected }\}$ as the subspace of continua subset of X. Denote by $f_*\colon \mathbb{K} \to \mathbb{K}$ the homeomorphism induced by f on the compact subsets of X with the Hausdorff metric. The induced action on continua will be denoted as $f_c\colon C(X) \to C(X)$. Note that f_c is a restriction of f_*.

We consider the four possibilities shown in Table (4.3). The implications of the table are easy to prove.

$$\begin{array}{ccc} f_* \text{ is expansive} & \Rightarrow & f_* \text{ is cw-expansive} \\ \Downarrow & & \Downarrow \\ f_c \text{ is expansive} & \Rightarrow & f_c \text{ is cw-expansive.} \end{array} \tag{4.3}$$

Remark 4.2.1. *If f_c is cw-expansive then* $\dim(X) \leq 2$. *This is because, by Theorem 4.1.17 we known that* $\dim(C(X)) < \infty$. *And it is known that if* $\dim(X) \geq 3$ *then* $\dim(C(X)) = \infty$ *(see* [92] *Theorem 2.3 and Theorem 2.5 for some results in the case* $\dim(X) = 2$*).*

Proposition 4.2.2. *It holds that f_* is cw-expansive if and only if* $\dim(X) = 0$.

Proof. If $\dim(X) = 0$ then $\dim(\mathbb{K}) = 0$. Therefore, $f_* \colon \mathbb{K} \to \mathbb{K}$ is cw-expansive. Conversely, if f_* is cw-expansive then $\dim(\mathbb{K})$ is finite. By [92] Theorem 1.95, we have that $\dim(X) = 0$. □

Definition 4.2.3. A homeomorphism $f \colon X \to X$ on a compact metric space is *hyper-expansive* if f_* is expansive, that is, there is $\delta > 0$ such that if $\operatorname{dist}_H(f_*^n(A), f_*^n(B)) < \delta$ for all $n \in \mathbb{Z}$, with A and B compact subsets of X, then $A = B$.

Notice that hyper-expansiveness is a stronger condition than expansiveness.

Remark 4.2.4. *We have seen in Theorem 4.1.17 that if a compact metric space admits a cw-expansive homeomorphism then its topological dimension is finite. It is known that if* $\dim_{top} X > 0$ *then* $\dim_{top} \mathbb{K} = \infty$ *(this fact was first proved in* [84]*, see also* [92] *Theorem 1.95). Hence, if f_* is cw-expansive then* $\dim_{top} X = 0$.

It is known that expansiveness does not imply hyper-expansiveness. Indeed, in [13] it is noticed that the shift map is not hyper-expansive. This can be deduced from the fact that the shift map has infinitely many periodic points. Those remarks on hyper-expansiveness were rediscovered in [93] (Proposition 2.23 and Example 2.24). We will give a simple characterization of hyper-expansiveness in Theorem 4.2.10. We need some definitions and previous results.

Definition 4.2.5. Let us denote

- Ωf the set of non-wandering points, i.e., $x \in \Omega f$ if for all $\varepsilon > 0$ there is $n > 0$ such that $f^n(B_\varepsilon x) \cap B_\varepsilon x \neq \emptyset$,
- Per_r the set of repeller periodic points and Per_a the set of attracting periodic points.

Remark 4.2.6. *It is easy to see that every expansive homeomorphism has a finite number of fixed points. Also, every compact f-invariant set $K \subset X$ (i.e., $f(K) = K$) is a fixed point of f_*. So, if f is hyper-expansive then f has a finite number of compact invariant sets (in particular, it has finitely many periodic points).*

Lemma 4.2.7. *If $f \colon X \to X$ is hyper-expansive and $K \subset X$ is minimal then K is finite (i.e., a periodic orbit).*

Proof. Minimality implies that for all $\varepsilon > 0$ there is $n \geq 0$ such that for all $x \in X$ the set $O_n x = \{x, fx, \dots, f^n x\}$ is ε-dense in K (i.e., for all $y \in K$ there is $j \in \{0, 1, \dots, n\}$ such that $\operatorname{dist}(y, f^j x) < \varepsilon$). Therefore, $f^j(O_n x)$ is ε-dense for all $j \in \mathbb{Z}$ because $f^j(O_n x) = O_n(f^j x)$. If ε is an expansive constant for f_* then $\operatorname{dist}_H(f^j(O_n x), f^j(K)) < \varepsilon$ for all $j \in \mathbb{Z}$. Then $K = O_n x$ and K is finite. □

Remark 4.2.8. *In the previous proof the expansiveness was contradicted with two sets $K_1 \subset K_2$. Notice that $\operatorname{dist}_H(A, B) \geq \operatorname{dist}_H(A, B \cup A)$, so f is hyper-expansive if and only if there is $\delta > 0$ such that if $A \subset B$, $A, B \in \mathbb{K}$ and $\operatorname{dist}_H(f_*^n A, f_*^n B) < \delta$ for all $n \in \mathbb{Z}$ then $A = B$.*

We have that if f is hyper-expansive then f has a finite number of periodic points. Eventually taking a power of f we can suppose that every periodic point is a fixed point. Recall that if a homeomorphism is expansive then its non-trivial powers are expansive too.

Lemma 4.2.9. *If f is hyper-expansive then every fixed point of f is an attractor or a repeller.*

Proof. By contradiction suppose that p is a fixed point of f that is neither attractor nor repeller. Since p is not an attractor, p is not stable (Remark 3.4.4). So, there is $\varepsilon > 0$ and a sequence x_n such that $x_n \to p$ as $n \to \infty$ and for some $k_n > 0$, $f^{k_n}(x_n) \notin B_\varepsilon(p)$. Suppose that for all $k < k_n$, $f^k(x_n) \in B_\varepsilon(p)$. Assume that $a_n = f^{k_n - 1}(x_n)$ converges to $a \in \operatorname{clos} B_\varepsilon(p)$. It is easy to see that $f^j(a) \to p$ as $j \to -\infty$ and $a \neq p$.

Similarly, using that p is not unstable, one can prove that there is $b \neq p$ such that $f^j(b) \to p$ as $j \to +\infty$. Let $\delta > 0$ be an expansive constant for f_*. Take $n_0 \geq 0$ such that $f^m(b), f^{-m}(a) \in B_\delta(p)$ for all $m \geq n_0$. Let $A = \{f^n(b), f^{-n}(a)\}$ and $B = A \cup \{p\}$. So, $A \neq B$ and $\operatorname{dist}_H(f^n A, f^n B) < \delta$ for all $n \in \mathbb{Z}$. That contradicts the expansiveness of f_*. □

Theorem 4.2.10. *A homeomorphism $f \colon X \to X$ is hyper-expansive if and only if f has a finite number of orbits and $\Omega f = Per_r \cup Per_a$.*

Proof. Direct. Suppose that f is hyper-expansive. We have proved that there is a finite number of periodic points. So, eventually taking a power of f we can suppose that every periodic point is in fact a fixed point. If there are only fixed points, there is nothing to prove (X is finite). So, suppose that $x \in X$ is not a fixed point. Consider the ω-limit set $\omega(x)$. It is a compact invariant set, therefore it contains a minimal set, say K. We have proved that every minimal set is a periodic orbit, so, it is a fixed point $K = \{p\}$. It is easy to see that $\omega(x) = \{p\}$, since p must be an attractor. In particular x is a wandering point. Then, we have proved that $\Omega(f) = \mathrm{Per}_a \cup \mathrm{Per}_r$.

Now we will prove that there is a finite number of orbits. It is easy to see that for all $\varepsilon > 0$ there is $N \geq 0$ such that if $x \notin B_\varepsilon(\Omega(f))$ then $f^j x, f^k x \in B_\varepsilon(\Omega(f))$ if $j \leq -N$ and $k \geq N$.

If f has an infinite number of orbits and $\varepsilon > 0$ is smaller than an expansive constant for f_*, then $X \setminus B_\varepsilon(\Omega(f))$ is a compact infinite set. So, there are $x, y \notin B_\varepsilon(\Omega(f))$ and $p, q \in \Omega(f)$ such that $\omega(x) = \omega(y) = \{p\}$ and $\alpha(x) = \alpha(y) = \{q\}$. Then, if $\mathrm{dist}(x, y)$ is small, this two points contradicts the expansiveness of f (and hyper-expansiveness too). This contradiction proves that there is a finite number of orbits.

Converse. Again, eventually taking a power of f, we can assume that every periodic point of f is a fixed point. Let $\delta_1 > 0$ be such that $\cap_{n\geq 0} f^n(B_{\delta_1}(\mathrm{Per}_a)) = \mathrm{Per}_a$ and $\cap_{n\leq 0} f^n(B_{\delta_1}(\mathrm{Per}_r)) = \mathrm{Per}_r$. Take $x_1, \dots, x_n$ one point of each wandering orbit of f. Let $\delta_2 > 0$ be such that $B_{\delta_2}(x_i) = \{x_i\}$ for all $i = 1, \dots, n$. We will show that $\delta = \min\{\delta_1, \delta_2\}$ is an expansive constant for f_*. Let A, B be two compact sets such that $\mathrm{dist}_H(f^n A, f^n B) < \delta$ for all $n \in \mathbb{Z}$. If there is a wandering point x such that $x \in A \setminus B$ then there is $k \in \mathbb{Z}$ and $i \in \{1, \dots, n\}$ such that $f^k x = x_i$. So, $\mathrm{dist}_H(f^k A, f^k B) > \delta_2$. This contradiction proves that the wandering points of A and B coincide. If $A \neq B$ then there is a fixed point $p \in A \setminus B$ (similarly for $p \in B \setminus A$). Without loss of generality suppose that p is a repeller. Since $p \notin B$ then there is $\varepsilon > 0$ such that $B_\varepsilon(p) \cap B = \emptyset$. Take n such that $B_{\delta_1}(p) \cap f^n B = \emptyset$. Since $p \in f^n A$ for all $n \in \mathbb{Z}$, we have that $\mathrm{dist}_H(f^n A, f^n B) > \delta_1$, which is a contradiction. So f is hyper-expansive. □

As we said, an important problem is to determine what spaces admit expansive homeomorphisms. In [63] this problem is solved for countable compact spaces. Now we give a characterization of compact spaces admitting hyper-expansive homeomorphisms.

A simple consequence of the previous result is that if X admits a hyper-expansive homeomorphism then X is countable. As we will see, the converse is not true. Let

$$\mathrm{Iso}(X) = \{x \in X : \text{there is } \varepsilon > 0 \text{ such that } B_\varepsilon(x) \cap X = \{x\}\}$$

and

$$\mathrm{Lim}(X) = X \setminus \mathrm{Iso}(X).$$

The cardinality of a set A is denoted as $|A|$.

Theorem 4.2.11. *A compact metric space X admits a hyper-expansive homeomorphism if and only if $2 \leq |\mathrm{Lim}(X)| < \infty$ or $\mathrm{Lim}(X) = \emptyset$ (i.e., X is finite).*

Proof. By Theorem 4.2.10 we have that $\mathrm{Lim}(X) \subset \Omega(f)$ that is because wandering points must be isolated. So, $\mathrm{Lim}(X)$ is finite. If X is infinite, there must be at least one attractor and one repeller, so $\mathrm{Lim}(X) \geq 2$.

In order to prove the converse notice that if the set of limit points is finite then X is countable. Consider an infinite countable space X (the finite case is trivial). Since every infinite continuum is uncountable, we have that $\dim_{top}(X) = 0$. It is known that if $\dim_{top}(X) \leq n$ then X is homeomorphic to a compact subset of $\mathbb{R}^{2n+1}$, see Theorem V2 in [54]. So, without loss of generality, we can assume that $X \subset \mathbb{R}$. Let $p_1 < \cdots < p_n \in X$, $n \geq 2$, be the limit points of X. We can also suppose that $X \subset [p_1, p_n]$ and for all $\varepsilon > 0$ we have that

- $X \cap (p_j, p_j + \varepsilon) \neq \emptyset$ for all $j = 1, \dots, n-1$ and
- $X \cap (p_j - \varepsilon, p_j) \neq \emptyset$ for all $j = 2, \dots, n$

Define $I_j = X \cap (p_j, p_{j+1})$ for $j = 1 \dots, n-1$. Now we define $f\colon X \to X$ as follows:

- $f(p_j) = p_j$ for all $j = 1, \dots, n$,
- if $x \in I_j$ and j is odd then $f(x)$ is the first point of X at the right of x and
- if $x \in I_j$ and j is even then $f(x)$ is the first point of X at the left of x.

In this way p_j is a repeller fixed point if j is odd and it is an attractor if j is even. So, by Theorem 4.2.10 we have that f is hyper-expansive. □

Since hyper-expansiveness is a very strong condition, we have that *most* homeomorphisms satisfy the following result.

Corollary 4.2.12. *If $f\colon X \to X$ is a homeomorphism of a compact metric space X and $|\operatorname{Lim}(X)| = \infty$ then for all $\varepsilon > 0$ there are two different compact sets $A, B \subset X$ such that*

$$\operatorname{dist}_H(f^n A, f^n B) < \varepsilon, \text{ for all } n \in \mathbb{Z}.$$

It is a simple consequence of our previous result. It holds for example if X is a manifold of positive dimension, a non-trivial connected space or a Cantor set.

Let us now give some examples and remarks.

Example 4.2.13. *Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$. Since X has just one limit point we have that X does not admit hyper-expansive homeomorphisms, but it is easy to see that it admits an expansive one.*

Countable compact spaces admitting expansive homeomorphisms can be characterized as follows. Recall that $\operatorname{Lim}^{\lambda+1}(X) = \operatorname{Lim}(\operatorname{Lim}^\lambda(X))$, $\operatorname{Lim}^1(X) = \operatorname{Lim}(X)$ and

$$\operatorname{Lim}^\lambda(X) = \bigcap_{\alpha<\lambda} \operatorname{Lim}^\alpha(X)$$

for every limit ordinal number λ. The *limit degree* of X is the ordinal number $d(X) = \lambda$ if $\operatorname{Lim}^\lambda(X) \neq \emptyset$ and $\operatorname{Lim}^{\lambda+1}(X) = \emptyset$. In [63] (Theorem 2.2) it is shown that a countable compact space X admits an expansive homeomorphism if and only if $d(X)$ is not a limit ordinal number.

Remark 4.2.14. *Applying Theorem 4.2.11 we have that X admits a hyper-expansive homeomorphism if and only if $d(X) \leq 1$ and $|\operatorname{Lim}(X)| \neq 1$.*

It seems to be of interest to provide an example of a countable compact space do not admitting expansive homeomorphisms.

Example 4.2.15. *Given $A \subset \mathbb{R}$ we say that $(a,b) \in A \times A$ is an* adjacent pair *if there are no points of A in the open interval (a,b). The set of adjacent pairs is denoted as*

$$\mathrm{Adj}(A) = \{(a,b) \in A \times A : a < b, (a,b) \cap A = \emptyset\}.$$

Let $A_0 = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ and

$$A_{n+1} = A_n \cup \bigcup_{(a,b)\in \mathrm{Adj}(A_n \cap [0,1/n])} \{a + (b-a)/m : m \in \mathbb{N}\}.$$

Define $X = \cup_{n=0}^{\infty} A_n$. It is easy to see that it is a compact set and it is countable by construction. Notice that $d(X)$ is the first infinite ordinal number and therefore it is a limit ordinal number. To see that it does not admit an expansive homeomorphism notice that if $f\colon X \to X$ is a homeomorphism then $\mathrm{Lim}^\lambda(X)$ is an f-invariant set for all ordinal number λ. Now notice that for all $\varepsilon > 0$ there is a finite ordinal number λ such that $\mathrm{Lim}^\lambda(X) \subset [0,\varepsilon]$ and $\mathrm{Lim}^\lambda(X)$ is an infinite set. Therefore, every pair of different points $x,y \in \mathrm{Lim}^\lambda(X)$ contradict the ε-expansiveness of f. Since ε is arbitrary we have that X does not admit expansive homeomorphisms.

4.3 Observable cardinality

As we saw, there are several variations of the definition of expansive homeomorphism. Some variations of this definition are weaker, as for example continuum-wise expansiveness and N-expansiveness (recall Definitions 3.4.5 and 3.4.11). Other variations of expansiveness can be found in [18, 85, 107]. A branch of research in topological dynamics investigates the possibility of extending known results for expansive homeomorphisms to these versions. See for example [86, 97, 112, 117]. In Chapter 5 we will consider N-expansive homeomorphisms of surfaces.

Other related definitions are stronger than expansiveness as for example positive expansiveness and hyper-expansiveness (recall Definitions 3.2.1 and 4.2.3). Both definitions are so strong that their examples are almost trivial. Recall Theorems 3.2.3 and 4.2.10.

These results give us general results about the dynamic of most homeomorphisms. We have that if the compact metric space X is not a finite set, then for every homeomorphism $f\colon X \to X$ and for all $\delta > 0$ there are $x \neq y$ such that $\mathrm{dist}(f^k(x), f^k(y)) < \delta$ for all $k \geq 0$. This is a very general result about the dynamics of homeomorphisms of compact metric spaces. We also have that no uncountable compact metric space admits a hyper-expansive homeomorphism. Therefore, if X is an uncountable compact metric space, as for example a compact manifold, then for every homeomorphism $f\colon X \to X$ and for all $\delta > 0$ there are two compact subsets $A, B \subset X$, $A \neq B$, such that $\mathrm{dist}_H(f^k(A), f^k(B)) < \delta$ for all $k \in \mathbb{Z}$. Recall the Hausdorff distance in Definition 3.4.7.

According to Lewowicz [75] we can explain the meaning of expansiveness as follows. Let us say that a *δ-observer* is someone that cannot distinguish two points if their distance is smaller than δ. If $\text{dist}(x, y) < \delta$ a δ-observer will not be able to say that the set $A = \{x, y\}$ has two points. But if the homeomorphism is expansive, with expansive constant greater than δ, and if the δ-observer knows all of the iterates $f^k(A)$ with $k \in \mathbb{Z}$, then he will find that A contains two different points, because if $\text{dist}(f^k(x), f^k(y)) > \delta$ then he will see two points in $f^k(A)$. Let us be more precise.

Definition 4.3.1. For $\delta \geq 0$, a set $A \subset X$ is *δ-separated* if for all $x \neq y$, $x, y \in A$, it holds that $\text{dist}(x, y) > \delta$. The *$\delta$-cardinality* of a set A is

$$|A|_\delta = \sup\{|B| : B \subset A \text{ and } B \text{ is } \delta\text{-separated}\},$$

where $|B|$ denotes the cardinality of the set B.

Notice that the δ-cardinality is always finite because X is compact. The δ-cardinality of a set represents the maximum number of different points that a δ-observer can identify in the set.

In this section we introduce a series of definitions, some weaker and other stronger than expansiveness, extending the notion of N-expansiveness of Morales. In terms of our δ-observer we can say that f is N-expansive if there is $\delta > 0$ such that if $|A| = N + 1$, a δ-observer will be able to say that A has at least two points given that he knows all of the iterates $f^k(A)$ for $k \in \mathbb{Z}$, i.e., $|f^k(A)|_\delta > 1$ for some $k \in \mathbb{Z}$. Let us introduce our main definition of this section.

Definition 4.3.2. Given integer numbers $m > n \geq 1$ we say that $f\colon X \to X$ is *(m, n)-expansive* if there is $\delta > 0$ such that if $|A| = m$ then there is $k \in \mathbb{Z}$ such that $|f^k(A)|_\delta > n$.

The first problem under study is the classification of these definitions. We prove that (m, n)-expansiveness implies N-expansiveness if $m \leq (N + 1)n$. In particular, if $m \leq 2n$ then (m, n)-expansiveness implies expansiveness. These results are stated in Corollary 4.3.9. We also show that (m, n)-expansiveness does not imply expansiveness if $n \geq 2$. For example, Anosov diffeomorphisms are known to be expansive and a consequence of Theorem 4.3.20 is that Anosov diffeomorphisms are not (m, n)-expansive for all $n \geq 2$.

It is a fundamental problem in dynamical systems to determine which spaces admit expansive homeomorphisms (or Anosov diffeomorphisms). In this paper we prove that no Peano continuum admits a (m, n)-expansive homeomorphism if $2m \geq 3n$, see Theorem 4.3.15. We also show that if X admits a $(n+1, n)$-expansive homeomorphism with $n \geq 3$ then X is a finite set. Examples of $(3, 2)$-expansive homeomorphisms are given on countable spaces (hyper-expansive homeomorphisms), see Theorem 4.3.17.

Our results related with (m, n)-expansiveness are organized as follows. In Section 4.3.1 we prove basic properties of (m, n)-expansive homeomorphisms. In Section 4.3.2 we prove that no infinite compact metric space admits a $(4, 3)$-expansive homeomorphism. In Section 4.3.3

we show that no Peano continuum admits a (m,n)-expansive homeomorphism if $2m \geq 3n$. In Section 4.2 we show that hyper-expansive homeomorphisms are $(3,2)$-expansive. In Section 4.3.5 we prove that a homeomorphism with the shadowing property and with two points x, y satisfying

$$0 = \liminf_{k\to\infty} \operatorname{dist}(f^k(x), f^k(y)) < \limsup_{k\to\infty} \operatorname{dist}(f^k(x), f^k(y))$$

cannot be $(m,2)$-expansive if $m > 2$.

4.3.1 Separating Finite Sets

Let (X, dist) be a compact metric space and consider a homeomorphism $f\colon X \to X$. Let us recall that for integer numbers $m > n \geq 1$ a homeomorphism f is (m,n)-*expansive* if there is $\delta > 0$ such that if $|A| = m$ then there is $k \in \mathbb{Z}$ such that $|f^k(A)|_\delta > n$. In this case we say that δ is a (m,n)-*expansive constant*. The idea of (m,n)-expansiveness is that our δ-observer will find more than n points in every set of m points if he knows all of its iterates.

Remark 4.3.3. *From the definitions it follows that a homeomorphism is* $(N+1,1)$*-expansive if and only if it is* N*-expansive. In particular,* $(2,1)$*-expansiveness is equivalent with expansiveness.*

Remark 4.3.4. *Notice that if* X *is a finite set then every homeomorphism of* X *is* (m,n)*-expansive.*

Proposition 4.3.5. *If* $n' \leq n$ *and* $m - n \leq m' - n'$ *then* (m,n)*-expansive implies* (m',n')*-expansive with the same expansive constant.*

Proof. The case $|X| < \infty$ is trivial, so, let us assume that $|X| = \infty$. Consider $\delta > 0$ as a (m,n)-expansive constant. Given a set A with $|A| = m'$ we will show that there is $k \in \mathbb{Z}$ such that $|f^k(A)|_\delta > n'$, i.e., the same expansive constant works. We divide the proof in two cases.

First assume that $m' \geq m$. Let $B \subset A$ with $|B| = m$. Since f is (m,n)-expansive, there is $k \in \mathbb{Z}$ such that $|f^k(B)|_\delta > n$. Therefore $|f^k(A)|_\delta > n \geq n'$, proving that f is (m',n')-expansive.

Now suppose that $m' < m$. Given that $|A| = m'$ and $|X| = \infty$ there is $C \subset X$ such that $A \cap C = \emptyset$ and $|A \cup C| = m$. By (m,n)-expansiveness, there is $k \in \mathbb{Z}$ such that $|f^k(A \cup C)|_\delta > n$. Then, there is a δ-separated set $D \subset f^k(A \cup C)$ with $|D| > n$. Notice that

$$|f^k(A) \cap D| = |D \setminus f^k(C)| \geq |D| - |f^k(C)| > n - (m - m')$$

and since $n - (m - m') \geq n'$ by hypothesis, we have that $f^k(A) \cap D$ is a δ-separated subset of $f^K(A)$ with more than n' points, that is $|f^k(A)|_\delta > n'$. This proves the $(m'n')$-expansiveness of f in this case too. □

As a consequence of Proposition 4.3.5 we have that

1. (m, n)-expansive implies $(m+1, n)$-expansive and
2. (m, n)-expansive implies $(m-1, n-1)$-expansive.

In Table 4.3.1 below we can easily see all these implications. The following proposition allows us to draw more arrows in this table, for example: $(4, 2) \Rightarrow (2, 1)$.

Table 4.1: Basic hierarchy of (m, n)-expansiveness. Each pair (m, n) in the table stands for "(m, n)-expansive". In the first position, (2,1), we have expansiveness. In the first line, of the form $(N+1, 1)$, we have N-expansive homeomorphisms.

$$\begin{array}{ccccccc}
(2,1) & \Rightarrow & (3,1) & \Rightarrow & (4,1) & \Rightarrow & \dots \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
(3,2) & \Rightarrow & (4,2) & \Rightarrow & (5,2) & \Rightarrow & \dots \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
(4,3) & \Rightarrow & (5,3) & \Rightarrow & (6,3) & \Rightarrow & \dots \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
\dots & & \dots & & \dots & &
\end{array}$$

Proposition 4.3.6. *If $a, n \geq 2$ and $f : X \to X$ is an (an, n)-expansive homeomorphism then f is $(a, 1)$-expansive.*

In order to prove it, let us introduce two previous results.

Lemma 4.3.7. *If $A, B \subset X$ are finite sets and $\delta > 0$ satisfies $|A| = |A|_\delta$ and $|B|_\delta = 1$ then for all $\varepsilon > 0$ it holds that*

$$|A \cup B|_{\delta+\varepsilon} \leq |A|_\varepsilon + |B|_\delta - |A \cap B|.$$

Proof. If $A \cap B = \emptyset$ then the proof is easy because

$$|A \cup B|_{\delta+\varepsilon} \leq |A|_{\delta+\varepsilon} + |B|_{\delta+\varepsilon} \leq |A|_\varepsilon + |B|_\delta.$$

Assume now that $A \cap B \neq \emptyset$. Since $|A| = |A|_\delta$ we have that A is δ-separated. Therefore $|A \cap B| = 1$ because $|B|_\delta = 1$. Assume that $A \cap B = \{y\}$. Let us prove that $|A \cup B|_{\delta+\varepsilon} \leq |A|_\varepsilon$ and notice that it is sufficient to conclude the proof of the lemma.

Let $C \subset A \cup B$ be a $(\delta+\varepsilon)$-separated set such that $|C| = |A \cup B|_{\delta+\varepsilon}$. If $C \subset A$ then

$$|A \cup B|_{\delta+\varepsilon} = |A|_{\delta+\varepsilon} \leq |A|_\varepsilon.$$

Therefore, let us assume that there is $x \in C \setminus A$. Define the set

$$D = (C \cup \{y\}) \setminus \{x\}.$$

Notice that $|C| = |D|$ and $D \subset A$.

We will show that D is ε-separated. Take $p, q \in D$ and arguing by contradiction assume that $p \neq q$ and $\text{dist}(p, q) \leq \varepsilon$. If $p, q \in C$ there is nothing to prove because C is $(\delta + \varepsilon)$-separated. Assume now that $p = y$. We have that $\text{dist}(x, p) \leq \delta$ because $x, p \in B$ and $|B|_\delta = 1$. Thus

$$\text{dist}(x, q) \leq \text{dist}(x, p) + \text{dist}(p, q) \leq \varepsilon + \delta.$$

But this is a contradiction because $x, q \in C$ and C is $(\varepsilon + \delta)$-separated. □

Lemma 4.3.8. *If f is $(m + l, n + 1)$-expansive then f is (m, n)-expansive or $(l, 1)$-expansive.*

Proof. Let us argue by contradiction and take an $(m + l, n + 1)$-expansive constant $\alpha > 0$. Since f is not (m, n)-expansive for $\varepsilon \in (0, \alpha)$ there is a set $A \subset X$ such that $|A| = m$ and $|f^k(A)|_\varepsilon \leq n$ for all $k \in \mathbb{Z}$. Take $\delta > 0$ such that $|A| = |A|_\delta$ and $\delta + \varepsilon < \alpha$.

Since f is not $(l, 1)$-expansive there is B such that $|B| = l$ and $|f^k(B)|_\delta = 1$ for all $k \in \mathbb{Z}$. By Lemma 4.3.7 we have that

$$|f^k(A \cup B)|_{\delta+\varepsilon} \leq |f^k(A)|_\varepsilon + |f^k(B)|_\delta - |A \cap B| \leq n + 1 - |A \cap B|,$$

for all $k \in \mathbb{Z}$. Also, we know that $|A \cup B| = m + l - |A \cap B|$. If we denote $r = |A \cap B|$ then f is not $(m + l - r, n + 1 - r)$-expansive. And by Proposition 4.3.5 we conclude that f is not $(m + l, n + 1)$-expansive. This contradiction proves the lemma. □

Proof of Proposition 4.3.6. Assume by contradiction that f is not $(a, 1)$-expansive. Since f is (an, n)-expansive, by Lemma 4.3.8 we have that f has to be $(a(n - 1), n - 1)$-expansive. Arguing inductively we can prove that f is $(a(n - j), n - j)$-expansive, for $j = 1, 2, \ldots, n - 1$. In particular, f is $(a, 1)$-expansive, which is a contradiction that proves the proposition. □

Corollary 4.3.9. *If $m \leq an$ and f is (m, n)-expansive then f is $(a, 1)$-expansive (i.e. $(a - 1)$-expansive). In particular, if $m \leq 2n$ and f is (m, n)-expansive then f is expansive.*

Proof. By Proposition 4.3.5 we have that f is (an, n)-expansive. Therefore, by Proposition 4.3.6 we have that f is $(a, 1)$-expansive. □

4.3.2 Separating 4 points

In this section we prove that $(n + 1, n)$-expansiveness with $n \geq 3$ implies that X is finite.

Theorem 4.3.10. *If X is a compact metric space admitting a $(4, 3)$-expansive homeomorphism then X is a finite set.*

Proof. By contradiction assume that f is a $(4,3)$-expansive homeomorphism of X with $|X| = \infty$ and take an expansive constant $\delta > 0$. We know that f cannot be positive expansive (recall Theorem 3.2.3). Therefore there are x_1, x_2 such that $x_1 \neq x_2$ and

$$\operatorname{dist}(f^k(x_1), f^k(x_2)) < \delta \tag{4.4}$$

for all $k \geq 0$. Analogously, f^{-1} is not positive expansive, and we can take y_1, y_2 such that $y_1 \neq y_2$ and

$$\operatorname{dist}(f^k(y_1), f^k(y_2)) < \delta \tag{4.5}$$

for all $k \leq 0$. Consider the set $A = \{x_1, x_2, y_1, y_2\}$. We have that $2 \leq |A| \leq 4$ (we do not know if the 4 points are different). By inequalities (4.4) and (4.5) we have that $|f^k(A)|_\delta < |A|$ for all $k \in \mathbb{Z}$. If $n = |A|$ then we have that f is not $(n, n-1)$-expansive. In any case, $n = 2, 3$ or 4, by Proposition 4.3.5 (see Table 4.1) we conclude that f is not $(4,3)$-expansive. This contradiction finishes the proof. □

Remark 4.3.11. *If X is a compact metric space admitting a $(n+1, n)$-expansive homeomorphism with $n \geq 3$ then X is a finite set. It follows by Theorem 4.3.10 and Proposition 4.3.5.*

Corollary 4.3.12. *If $f\colon X \to X$ is a homeomorphism of a compact metric space and $|X| = \infty$ then for all $\delta > 0$ and $m \geq 4$ there is $A \subset X$ with $|A| = m$ such that $|f^k(A)|_\delta < |A|$ for all $k \in \mathbb{Z}$.*

Proof. It is just a restatement of Remark 4.3.11. □

4.3.3 On Peano continua

In this section we study (m,n)-expansiveness on Peano continua. Let us start recalling that a *continuum* is a compact connected metric space and a *Peano continuum* is a locally connected continuum. A singleton space ($|X| = 1$) is a *trivial* Peano continuum.

Definition 4.3.13. For $x \in X$ and $\delta > 0$ define the *stable* and *unstable* set of x as

$$W^s_\delta(x) = \{y \in X : \operatorname{dist}(f^k(x), f^k(y)) \leq \delta \,\forall\, k \geq 0\},$$
$$W^u_\delta(x) = \{y \in X : \operatorname{dist}(f^k(x), f^k(y)) \leq \delta \,\forall\, k \leq 0\}.$$

Remark 4.3.14. *Notice that (m,n)-expansiveness implies continuum-wise expansiveness for all $m > n \geq 1$.*

Theorem 4.3.15. *If X is a non-trivial Peano continuum then no homeomorphism of X is (m,n)-expansive if $2m \leq 3n$.*

Proof. Let δ be a positive real number and assume that f is (m,n)-expansive. As we remarked above, f is a continuum-wise expansive homeomorphism. By Proposition 4.1.8 we know that for such homeomorphisms on a Peano continuum, every point has non-trivial stable and unstable sets. Take n different points $x_1,\dots,x_n \in X$ and let $\delta' \in (0,\delta)$ be such that $\text{dist}(x_i,x_j) > 2\delta'$ if $i \neq j$. For each $i = 1,\dots,n$, we can take $y_i \in W^s_{\delta'}(x_i)$ and $z_i \in W^u_{\delta'}(x_i)$ with $x_i \neq y_i$ and $x_i \neq z_i$. Consider the set $A = \{x_1,y_1,z_1,\dots,x_n,y_n,z_n\}$. Since $\text{dist}(x_i,x_j) > 2\delta'$ if $i \neq j$, and $y_i, z_i \in B_{\delta'}(x_i)$ we have that $|A| = 3n$. If A_i denotes the set $\{x_i,y_i,z_i\}$ we have that $|f^k(A_i)|_{\delta'} \leq 2$ for all $k \in \mathbb{Z}$. This is because if $k \geq 0$ then $\text{dist}(f^k(x_i),f^k(y_i)) \leq \delta'$ and if $k \leq 0$ then $\text{dist}(f^k(x_i),f^k(z_i)) \leq \delta'$. Therefore $|f^k(A)|_{\delta'} \leq 2n$, and then $|f^k(A)|_\delta \leq 2n$. Since $\delta > 0$ and $n \geq 1$ are arbitrary, we have that f is not $(3n,2n)$ expansive for all $n \geq 1$. Finally, by Proposition 4.3.5, we have that f is not (m,n)-expansive if $2m \leq 3n$. □

Corollary 4.3.16. *If $f\colon X \to X$ is a homeomorphism and X is a non-trivial Peano continuum then for all $\delta > 0$ there is $A \subset X$ such that $|A| = 3$ and $|f^k(A)|_\delta \leq 2$ for all $k \in \mathbb{Z}$.*

Proof. By Theorem 4.3.15 we know that f is not $(3,2)$-expansive. Therefore, the proof follows by definition. □

4.3.4 Hyper-expansive homeomorphisms

Theorem 4.3.17. *If $f\colon X \to X$ is a hyper-expansive homeomorphism and $|X| = \infty$ then f is (m,n)-expansive if and only if $n < 3$.*

Proof. Let us start with the converse part of the theorem. We know that f is $(m,1)$-expansive for all $m > 1$. We will show that f is $(3,2)$-expansive (which implies $(m,2)$-expansivity for all $m > 2$). Let P_a be the set of periodic attractors, P_r the set of periodic repellers and take $x_1,\dots,x_j$ one point in each wandering orbit (recall that, Theorem 4.2.10, hyper-expansiveness implies that f has just a finite number of orbits). Define $Q = \{x_1,\dots,x_j\}$. Take $\delta > 0$ such that

1. if $p,q \in P_a \cup P_r$ and $p \neq q$ then $\text{dist}(p,q) > \delta$,
2. if $x_i \in Q$ then $B_\delta(x_i) = \{x_i\}$, recall that wandering points are isolated points by Theorem 4.2.10,
3. if $p \in P_a$, $x_i \in Q$ and $k \leq 0$ then $\text{dist}(p,f^k(x_i)) > \delta$,
4. if $q \in P_r$, $x_i \in Q$ and $k \geq 0$ then $\text{dist}(p,f^k(x_i)) > \delta$ and
5. if $x,y \in Q$ and $k > 0 > l$ then $\text{dist}(f^k(x),f^l(y)) > \delta$.

Let us prove that such δ is a $(3,2)$-expansive constant. Take $a,b,c \in X$ with $|\{a,b,c\}| = 3$. The proof is divided by cases:

- If $a,b,c \in P = P_a \cup P_r$ then item 1 above concludes the proof.

- If $a, b \in P$ and $c \notin P$ then there is $k \in \mathbb{Z}$ such that $f^k(c) \in Q$. In this case items 1 and 2 conclude the proof.
- Assume now that $a \in P$ and $b, c \notin P$. Without loss of generality let us suppose that a is a repeller. Let $k_b, k_c \in \mathbb{Z}$ be such that $f^{k_b}(b), f^{k_c}(c) \in Q$. Define $k = \min\{k_b, k_c\}$. In this way: $\text{dist}(f^k(a), f^k(b)), \text{dist}(f^k(a), f^k(c)) \geq \delta$ by item 4 and $\text{dist}(f^k(b), f^k(c)) \geq \delta$ by item 2.
- If $a, b, c \notin P$ then consider $k_a, k_b, k_c \in \mathbb{Z}$ such that $f^{k_a}(a), f^{k_b}(b), f^{k_c}(c) \in Q$. Assume, without loss, that $k_a \leq k_b \leq k_c$. Take $k = k_b$. In this way, items 2 and 5 finishes the converse part of the proof of the theorem.

To prove the direct part by contradiction, assume that f is (m', n)-expansive with $n \geq 3$. This implies that f is $(m, 3)$-expansive if $m' - m = n - 3$. Take $\delta > 0$ an $(m, 3)$-expansive constant. Since $|X| = \infty$ there is at least one wandering point x. Without loss of generality assume that $\lim_{k\to\infty} f^k(x) = p_a$ an attractor fixed point and $\lim_{k\to-\infty} f^k(x) = p_r$ a repeller fixed point. Take $k_1, k_2 \in \mathbb{Z}$ such that $\text{dist}(f^k(x), p_r) < \delta$ for all $k \leq k_1$ and $\text{dist}(f^k(x), p_a) < \delta$ for all $k \geq k_2$. Let $l = k_2 - k_1$ and define $x_1 = f^{-k_1}(x)$, and $x_{i+1} = f^l(x_i)$ for all $i \geq 1$. Consider the set $A = \{x_1, \dots, x_m\}$. By construction we have that $|A| = m$ and $|f^k(A)|_\delta \leq 3$ for all $k \in \mathbb{Z}$. Contradicting that δ is an $(m, 3)$-expansive constant and finishing the proof. □

Remark 4.3.18. *In light of the previous proof one may wonder if a* smart *δ-observer will not be able to say that A has more than 3 points. We mean, we are assuming that a δ-observer will say that A has n' points with*

$$n' = \max_{k\in\mathbb{Z}} |f^k(A)|_\delta.$$

According to the dynamic of the set A in the previous proof, we guess that with more reasoning a smarter δ-observer will find that A has more than 3 points.

Theorem 4.3.17 gives us examples of $(3, 2)$-expansive homeomorphisms on infinite countable compact metric spaces. A natural question is: does $(3, 2)$-expansiveness implies hyper-expansiveness? I do not know the answer, but let us remark some facts that may be of interest. If f is $(3, 2)$-expansive then:

- For all $x \in X$ either the stable or the unstable set must be trivial. It follows by the arguments of the proof of Theorem 4.3.15.
- If x, y are bi-asymptotic, i.e., $\text{dist}(f^k(x), f^k(y)) \to 0$ as $k \to \pm\infty$ then they are isolated points of the space. Suppose that x were an accumulation point. Given $\delta > 0$ take k_0 such that if $|k| > k_0$ then $\text{dist}(f^k(x), f^k(y)) < \delta$. Take a point z close to x such that $\text{dist}(f^k(x), f^k(z)) < \delta$ if $|k| \leq k_0$ (we are just using the continuity of f). Then x, y, z contradicts $(3, 2)$-expansiveness.

An expansive homeomorphism with these two properties was given in Section 3.7.1 (interval exchange subshift). This example is not hyper-expansive because, since it is minimal and non-trivial, the space is uncountable. But, is it $(3,2)$-expansive?

Proposition 4.3.19. *There are* $(4,2)$*-expansive homeomorphisms that are not* $(3,2)$*-expansive.*

Proof. Let us prove it giving an example. Consider a countable compact metric space X and a homeomorphism $f\colon X\to X$ with the following properties:

1. f has 5 orbits,
2. $a,b,c\in X$ are fixed points of f,
3. there is $x\in X$ such that $\lim_{k\to-\infty} f^k(x)=a$ and $\lim_{k\to+\infty} f^k(x)=b$,
4. there is $y\in X$ such that $\lim_{k\to-\infty} f^k(y)=b$ and $\lim_{k\to+\infty} f^k(y)=c$.

In order to see that f is not $(3,2)$-expansive consider $\varepsilon>0$. Take $k_0\in\mathbb{Z}$ such that for all $k\geq k_0$ it holds that $\operatorname{dist}(f^k(x),b)<\varepsilon$ and $\operatorname{dist}(f^{-k}(y),b)<\varepsilon$. Define $u=f^{k_0}(x)$ and $v=f^{-k_0}(y)$. In this way $|\{f^k(u),b,f^k(v)\}|_\varepsilon\leq 2$ for all $k\in\mathbb{Z}$. This proves that f is not $(3,2)$-expansive.

Let us now indicate how to prove that f is $(4,2)$-expansive. Consider $\varepsilon>0$ such that if $i\geq 0$ and $j\in\mathbb{Z}$ then $\operatorname{dist}(f^{-i}(x),f^j(y))>\varepsilon$ and $\operatorname{dist}(f^j(x),f^i(y))>\varepsilon$. Now, a similar argument as in the proof of Theorem 4.3.17, shows that f is $(4,2)$-expansive. □

4.3.5 With the shadowing property

In this section we prove that an important class of homeomorphisms are not (m,n)-expansive for all $m>n\geq 2$. In order to state this result let us recall that a δ*-pseudo orbit* is a sequence $\{x_k\}_{k\in\mathbb{Z}}$ such that $\operatorname{dist}(f(x_k),x_{k+1})\leq\delta$ for all $k\in\mathbb{Z}$. We say that a homeomorphism has the *shadowing property* if for all $\varepsilon>0$ there is $\delta>0$ such that if $\{x_k\}_{k\in\mathbb{Z}}$ is a δ-pseudo orbit then there is x such that $\operatorname{dist}(f^k(x),x_k)<\varepsilon$ for all $k\in\mathbb{Z}$. In this case we say that x ε*-shadows* the δ-pseudo orbit.

Theorem 4.3.20. *Let* $f\colon X\to X$ *be a homeomorphism of a compact metric space* X*. If* f *has the shadowing property and there are* $x,y\in X$ *such that*

$$0=\liminf_{k\to+\infty}\operatorname{dist}(f^k(x),f^k(y))<\limsup_{k\to+\infty}\operatorname{dist}(f^k(x),f^k(y))$$

then f *is not* (m,n)*-expansive.*

Proof. By Proposition 4.3.5 it is enough to prove that f cannot be $(m,2)$-expansive. Consider $\varepsilon>0$. We will define a set A with $|A|=\infty$ such that for all $k\in\mathbb{Z}$, $f^k(A)\subset B_\varepsilon(f^k(x))\cup B_\varepsilon(f^k(y))$, proving that f is not $(m,2)$-expansive for all $m>2$.

Consider two increasing sequences $a_l, b_l \in \mathbb{Z}$, $\rho \in (0,\varepsilon)$ and $\delta > 0$ such that

$$\begin{aligned} & a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < \dots, \\ & \text{dist}(f^{a_l}(x), f^{a_l}(y)) < \delta, \\ & \text{dist}(f^{b_l}(x), f^{b_l}(y)) > \rho \end{aligned}$$

for all $l \geq 1$ and assume that every δ-pseudo orbit can be $(\rho/2)$-shadowed. For each $l \geq 1$ define the δ-pseudo orbit z^l_k as

$$z^l_k = \begin{cases} f^k(x) \text{ if } k < a_l, \\ f^k(y) \text{ if } k \geq a_l. \end{cases}$$

Then, for every $l \geq 1$ there is a point w^l whose orbit $(\rho/2)$-shadows the δ-pseudo orbit $\{z^l_k\}_{k\in\mathbb{Z}}$. Let us now prove that if $1 \leq l < s$ then $w^l \neq w^s$. We have that $a_l < b_l < a_s$. Therefore $z^l_{b_l} = f^{b_l}(y)$ and $z^s_{b_l} = f^{b_l}(x)$. Since w^l and w^s $(\rho/2)$-shadows the pseudo orbits z^l and z^s respectively, we have that

$$\text{dist}(f^{b_l}(w^l), f^{b_l}(y)), \text{dist}(f^{b_l}(w^s), f^{b_l}(x)) < \rho/2.$$

We conclude that $w^l \neq w^s$ because $\text{dist}(f^{b_l}(x), f^{b_l}(y)) > \rho$. Therefore, if we define $A = \{w^l : l \geq 1\}$ we have that $|A| = \infty$. Finally, since $\rho < \varepsilon$, we have that $f^k(A) \subset B_\varepsilon(f^k(x)) \cup B_\varepsilon(f^k(y))$ for all $k \in \mathbb{Z}$. Therefore, $|f^k(A)|_\varepsilon \leq 2$ for all $k \in \mathbb{Z}$. □

Remark 4.3.21. *Examples where Theorem 4.3.20 can be applied are Anosov diffeomorphisms and symbolic shift maps. Also, if $f\colon X \to X$ is a homeomorphism with an invariant set $K \subset X$ such that $f\colon K \to K$ is conjugated to a symbolic shift map then Theorem 4.3.20 holds because the (m,n)-expansiveness of f in X implies the (m,n)-expansiveness of f restricted to any compact invariant set $K \subset X$ as can be easily checked.*

Chapter 5

Surface homeomorphisms

In this chapter we study cw-expansive homeomorphisms of compact surfaces. In Section 5.1 we construct product boxes for cw-expansive homeomorphisms without small bi-asymptotic sectors. Our proofs are based on Lewowicz ideas on the study of expansive surface homeomorphisms. In Section 5.2 we prove that 2-expansive homeomorphisms without wandering points are expansive. In Section 5.3 we give an example of a 2-expansive homeomorphism that is not expansive.

In this chapter S will be a compact surface and $f\colon S\to S$ will denote a homeomorphism.

5.1 Cw-expansiveness and bi-asymptotic sectors

5.1.1 Regular cw-expansiveness

Definition 5.1.1. A surface homeomorphism $f\colon S\to S$ is a *regular cw-expansive* homeomorphism if it is cw-expansive and stable and unstable continua are locally connected.

Recall from Definition 3.4.11, that f is N-*expansive* if there is $\delta>0$ such that if $\operatorname{diam}(f^n(C))\leq\delta$ for all $n\in\mathbb{Z}$ then C is a finite set.

Theorem 5.1.2. *Every N-expansive surface homeomorphism is a regular cw-expansive homeomorphism.*

Since this result is essentially [74, Lemma 2.3] or [52, Proposition 3.1], we only give a sketch.

Proof. Let $\alpha>0$ be such that if $\operatorname{diam}(f^n(A))<\alpha$ for all $n\in\mathbb{Z}$ then $|A|<\infty$. Arguing by contradiction assume that C is a stable continuum that is not locally connected at $x\in C$. Without loss of generality we assume that $\operatorname{diam}(f^n(C))<\alpha$ for all $n\geq 0$. Since C is not locally connected at x we have a sequence of continua X_k converging in the Hausdorff metric to a non-trivial continuum X_∞ with $k\to\infty$ such that $(X_\infty\cup_{k\in\mathbb{N}}X_k)\subset C$, $X_k\cap X_\infty=\emptyset$ and

$X_k \cap X_j = \emptyset$ if $j, k \in \mathbb{N}$ and $j \neq k$, (see Chapter IV [136]). Observe that there is an open set U that is separated by every X_k and by X_∞. Assume that $\operatorname{diam}(U) < \alpha$. Let $y_k \in X_k$ be such that $y_k \to y \in X_\infty \cap U$. Take $Y_k \subset \operatorname{clos}(U)$ an unstable continuum containing y_k meeting the boundary of U such that $\operatorname{diam}(f^n(Y_k)) < \alpha$ for all $n \leq 0$. Since f is N-expansive, if we let $k \to \infty$ we find that X_∞ contains a non-trivial continuum E that is a limit of Y_k. But this contradicts that f is cw-expansive and consequently that it is N-expansive. □

5.1.2 Bi-asymptotic sectors

Definition 5.1.3. A disc bounded by the union of a stable arc and an unstable arc is called a *bi-asymptotic sector*.

Proposition 5.1.4. [1] *Let f be a regular cw-expansive homeomorphism. Suppose that there is a small closed disc $U \subset S$ without bi-asymptotic sectors and containing an unstable arc α in its boundary. Then for all $x \in \alpha$ there is a stable arc from x to ∂U contained in U.*

Proof. Given $x \in \alpha$ denote by C the connected component of $W^s(x) \cap \operatorname{clos} U$ containing x. Let β be the arc $\partial U \setminus \alpha$. By contradiction assume that $C \cap \beta = \emptyset$. The point x separates α in two curves α_1 and α_2. Given $\varepsilon > 0$ consider a curve J starting at α_1 and ending in α_2 such that the interior of J is contained in $(U \setminus C) \cap B_\varepsilon(C)$. If ε is small we know that there is no stable arc from a point of J to β. Therefore, for every $z \in J$ the stable set of z meets the boundary of U at α. Let J_i be set of points of J such that its stable set cuts α_i, for $i = 1, 2$. Both sets are closed and non-empty. Also $J = J_1 \cup J_2$. Since J is connected, there is $p \in J_1 \cap J_2$. But this means that there is a stable arc from p to x, which is a contradiction because J is disjoint from C. □

5.1.3 Topological bi-asymptotic sectors

Let f be a cw-expansive homeomorphism.

Definition 5.1.5. Let C be a stable continuum and let D be an unstable continuum. We say that (C, D) form a *topological bi-asymptotic sector* if $C \cap D$ is not connected and $C \cup D$ is contained in a disc $U \subset S$.

Proposition 5.1.6. *If f is cw-expansive and has no arbitrarily small topological bi-asymptotic sectors then f is a regular cw-expansive homeomorphism.*

Proof. By contradiction assume that A is a stable continuum that is not locally connected. Let A_n, $n \geq 1$, be a sequence of pairwise disjoint stable subcontinua of A. Assume that A_n converges

[1] See Lemma 3.2 in [74]

to $A_* \subset A$ in the Hausdorff metric and A_n is disjoint from A_* for all $n \geq 1$. Take $p \in A_*$ and $\delta > 0$ such that A_* and A_n separate $B_\delta(p)$ for all $n \geq 1$ and there are no bi-asymptotic sectors in $B_\delta(p)$. Take $x_n \in A_n$ such that $x_n \to p$. For each n consider an unstable continuum U_n not contained in $B_\delta(p)$ such that $x_n \in U_n$. Since there are no topological bi-asymptotic sectors, each U_n can meet just one of the stable continua of the sequence A_n. Taking limit as $n \to \infty$ we obtain an unstable non-trivial subcontinuum contained in A_*. This contradicts that f is cw-expansive and the proof ends. □

5.1.4 Product boxes

Definition 5.1.7. A *product box* is a homeomorphism $\phi\colon [0,1] \times [0,1] \to K \subset S$ taking horizontal lines onto stable arcs and vertical lines onto unstable arcs. The *corners* and the *sides* of the product box are the images of the corners and the sides of the square $[0,1] \times [0,1]$.

Theorem 5.1.8. *If f is a regular cw-expansive surface homeomorphism without arbitrarily small bi-asymptotic sectors then for all $x \in S$ a neighborhood of x is covered by at least 4 product boxes with corner at x and intersecting in the sides.*

Proof. Given $x \in S$ consider an open disc D around x. Let α be a stable arc from x to ∂D. Consider an unstable arc β from x to ∂D such that there is a component U of $D \setminus (\alpha \cup \beta)$ such that there is neither a stable nor unstable arc from x to ∂D contained in U.

By Proposition 5.1.4 for each $y \in \alpha$ there is an unstable arc $u_y \subset U$ from y to ∂U. Also, for all $z \in \beta$ there is a stable arc $s_z \subset U$ from z to ∂U. Let us prove that there are $y_0 \in \alpha$ and $z_0 \in \beta$ such that u_{y_0} cuts s_{z_0}. If this is not the case we can take $y_n \in \alpha$ and $z_n \in \beta$ both converging to x and such that $u_{y_n} \cap s_{z_n} = \emptyset$. Then we have a contradiction by taking limit in the Hausdorff metric of the continua u_{y_n} (we obtain an unstable arc contained in U from x to ∂D).

Let V be the rectangle limited by the four curves α, β, u_{y_0} and s_{z_0}. Applying Proposition 5.1.4 in V it is easy to see that the closure of V is a product box. Now applying this construction a finite number of times the proof concludes. There will be at least 4 product boxes because there are no bi-asymptotic sectors. □

5.1.5 Expansive homeomorphisms

Theorem 5.1.9. *If $f\colon S \to S$ is a homeomorphism of a compact surface S then the following statements are equivalent:*

1. *f is expansive*
2. *f is cw-expansive without topological bi-asymptotic sectors.*
3. *f is a regular cw-expansive homeomorphism without bi-asymptotic sectors,*

Proof. $(1 \to 2)$. Expansive homeomorphisms of compact metric spaces are always cw-expansive. Also, expansive homeomorphisms of compact surfaces are known to be conjugate to pseudo-Anosov diffeomorphisms. Then, it is easy to see that there are no topological bi-asymptotic sectors.

$(2 \to 3)$. By Proposition 5.1.6 we have that f is a regular cw-expansive homeomorphism. It only rest to note that bi-asymptotic sectors are topological bi-asymptotic sectors.

$(3 \to 1)$. By Theorem 5.1.8 we can see that there is $\alpha > 0$ such that $W^s_\alpha(x) \cap W^u_\alpha(x) = \{x\}$ for all $x \in S$. Therefore, α is an expansive constant. □

5.2 Two-expansiveness

In this section $f: S \to S$ is a 2-expansive homeomorphisms with non-wandering set $\Omega(f) = S$. We will prove that such homeomorphism has no bi-asymptotic sectors (recall Definition 5.1.3). Then, applying our previous results we will be able to conclude that f is expansive.

Let $\alpha > 0$ be a 2-expansive constant for f, i.e., given any subset C of M, if $\operatorname{diam}(f^n(C)) \leq \alpha$ for all $n \in \mathbb{Z}$ then C has at most two points.

Let D be a bi-asymptotic sector of diameter less than α bounded by a stable arc a^s and an unstable arc a^u. For $p \in D$ define $C^s_D(p)$ and $C^u_D(p)$ as the connected component of $W^s(p) \cap D$ and $W^u(p) \cap D$ containing p respectively.

Lemma 5.2.1. *If $C^u_D(p)$ separates D then it meets twice the stable boundary a^s of D.*

Proof. Observe that D is a 2-disk. Fix p an interior point of D. Since $C^u_D(p)$ separates D we have that $\partial D \cap C^u_D(p)$ has at least two points. Moreover, since $C^u_D(p)$ is arc-connected, these two points can be joined by an arc b contained in $C^u_D(p)$. We need to show that these points are in a^s. There are three possible cases. In the first case b cuts twice the unstable boundary of D as in the first picture of Figure 5.1. Both unstable arcs bound an open disc U, as in the figure.

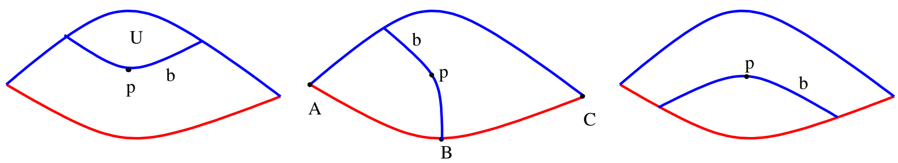

Figure 5.1: The only possible case is the right hand side picture.

This is a contradiction to Theorem 4.1.6, because the points in U are Lyapunov stable. The second case corresponds to b intersecting the stable and the unstable arcs of the bi-asymptotic sector. In this case we get three points at a^s contradicting the 2-expansiveness as shown in the second picture of Figure 5.1: the points A, B, C are in the same local stable and local unstable set. Therefore the only possible case corresponds to the right hand side picture at Figure 5.1, that is exactly what we want to prove. □

In the set $\mathcal{F}^s = \{C^s_D(x) : x \in D\}$ we can define an order as $C^s_D(x) < C^s_D(y)$ if a^s and $C^s_D(y)$ are separated by $C^s_D(x)$. See Figure 5.2.

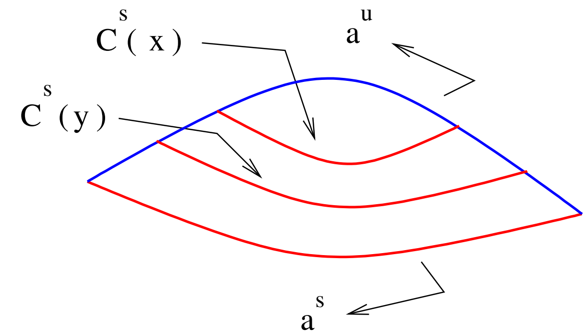

Figure 5.2: Order of stable arcs separating a bi-asymptotic sector.

Lemma 5.2.2. *The order* $<$ *in* $\mathcal{F}^s$ *is a total order.*

Proof. Given $C^s_D(x), C^s_D(y) \in \mathcal{F}^s$, $C^s_D(x) \neq C^s_D(y)$, we have to prove that $C^s_D(x) < C^s_D(y)$ or $C^s_D(y) < C^s_D(x)$. By contradiction assume this is not the case. Therefore we can consider $\gamma_1, \gamma_2, \gamma_3 \subset a^u$ three subarcs of the unstable boundary of the bi-asymptotic sector D. See Figure 5.3.

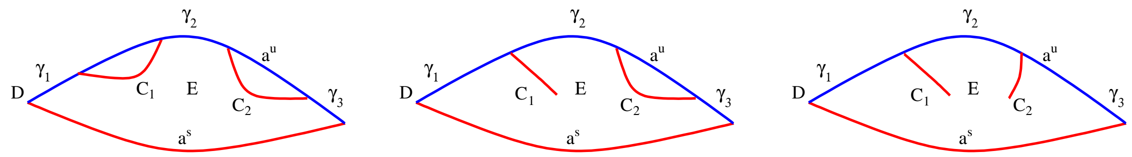

Figure 5.3: Impossible cases for a 2-expansive homeomorphism.

Let E be the connected component of $D \setminus (C^s_D(x) \cup C^s_D(y))$ containing a^s as shown in Figure 5.3. For $1 \leq i < j \leq 3$, define

$$A_{ij} = \{x \in E : C^s_D(x) \cap \gamma_i \neq \emptyset, C^s_D(x) \cap \gamma_j \neq \emptyset\}.$$

We have that $C^s_D(x) \subset A_{12}$, $C^s_D(y) \subset A_{23}$ and $a^s \subset A_{13}$, so, these sets are not empty. It is easy to see that they are closed and by the previous lemma they cover E. Since E is connected they can not be disjoint, but this contradicts 2-expansiveness. □

Given a stable arc b separating D we consider the map $g : b \to b$ defined by

$$C^u_D(x) \cap b = \{x, g(x)\}.$$

Notice that if $C^u_D(x)\cap C^s_D(x)=\{x\}$ then $g(x)=x$. The hypothesis of 2-expansiveness implies that $C^u_D(x)\cap b$ has at most two points, therefore g is well defined.

Lemma 5.2.3. *For every stable arc $b\subset D$ separating D, the map $g\colon b\to b$ is continuous.*

Proof. Since b is homeomorphic to the interval $[0,1]$ we can consider in b an order defining its topology. We will show that g is decreasing with respect to such an order on the arc b. It is well known that this allows us to conclude that g is continuous because $g\colon b\to b$ is bijective, in fact $g=g^{-1}$ as can be easily seen from the definition of g.

By contradiction suppose that g is not decreasing. Then there are $x,y\in b$ such that $x<y$ and $g(x)<g(y)$. We have essentially two possible cases: $x<g(x)<y<g(y)$ or $x<y<g(x)<g(y)$. Other cases are obtained interchanging x with $g(x)$ or y with $g(y)$. The first case contradicts Lemma 5.2.2 because the arc from x to $g(x)$ is not comparable with the arc from y to $g(y)$. The second case contradicts 2-expansiveness, because the unstable arc γ_1 from x to $g(x)$ and the arc γ_2 from y to $g(y)$ must have nontrivial intersection. Then $\gamma=\gamma_1\cup\gamma_2$ is a unstable continuum containing the four points $x,y,g(x),g(y)$. Since these points are also in the stable arc b we contradict 2-expansiveness. □

Lemma 5.2.4. *If $b\subset D$ is an unstable arc meeting twice a^s then there is $z\in b$ such that $b\cap C^s_D(z)=\{z\}$ (a fixed point of g).*

Proof. We need to prove that there is a fixed point of g in b as in Figure 5.4. Since b is

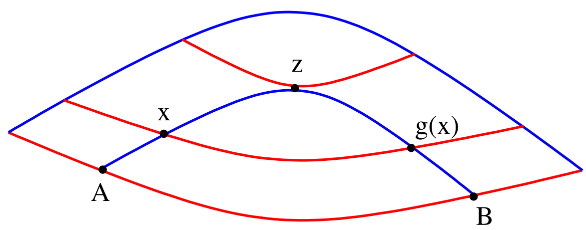

Figure 5.4: Illustration of the map g.

homeomorphic to an interval and g is a homeomorphism reversing orientation we have that g must have a fixed point $z\in b$. □

5.2.1 Regular bi-asymptotic sectors.

Definition 5.2.5. A bi-asymptotic sector is *regular* if for all p interior to D we have that $C^s_D(p)$ and $C^u_D(p)$ separate D.

Proposition 5.2.6. *If f is 2-expansive and $\Omega(f)=M$ then there are no regular bi-asymptotic sectors of diameter less than α.*

Proof. By contradiction assume that D is a regular bi-asymptotic sector of diameter smaller than α. Since there are no wandering points we have that there is $p \in D$ and $k > 0$ arbitrarily large such that $q = f^k(p)$ is in D. Since the set of points ξ such that $g(\xi) = \xi$ is of first category in the sense of Baire and $\Omega(f) = M$ there are (a residual subset of) points $p \in D$ such that $\{p, p'\} = C^s_D(p) \cap C^u_D(p)$ with $p \neq p'$. The points $\{p, p'\} = C^s_D(p) \cap C^u_D(p)$ determine a regular bi-asymptotic sector D_p contained in D as in Figure 5.5 (b). For arbitrarily large k, the stable

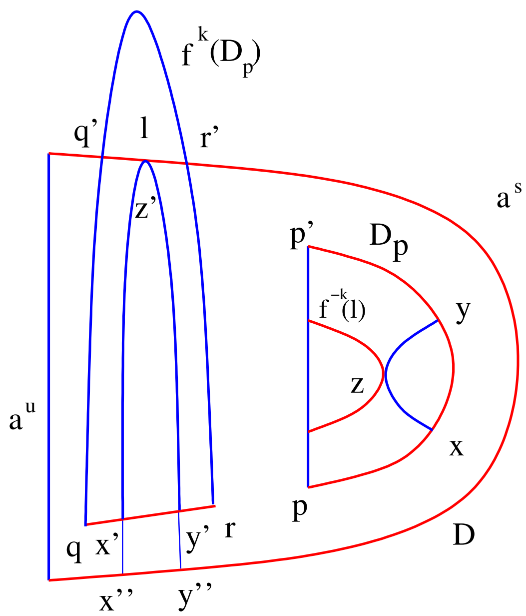

Figure 5.5: A regular bi-asymptotic sector.

arc (in red at Figure 5.5) defined by p and p' is transformed by f^k into a stable arc with extreme points $q = f^k(p)$ and $r = f^k(p')$ that is contained in D. The sector D_p is transformed into $f^k(D_p)$ and the image by f^k of the unstable arc $u(p, p')$ from p to p' is not contained in D. Then, there are two points $q', r' \in a^s \cap f^k(u(p, p'))$ as in Figure 5.5 (recall that a^s is the stable arc in the bi-asymptotic sector D). Now consider the stable arc $l = s(q', r')$ contained in a^s. The stable arc $f^{-k}(l)$ separates the bi-asymptotic sector D_p and therefore we can apply Lemma 5.2.4 to obtain a point $z \in f^{-k}(l)$ such that the unstable arc $u(z)$ through z in D_p meets $f^{-k}(l)$ only at z. Take x, y the intersection points of the stable arc in the boundary of D_p and $u(z)$. Consider the points $x' = f^k(x)$, $y' = f^k(y)$ and $z' = f^k(z)$ as in Figure 5.5. The unstable arcs in D through x' and y' meet a^s at x'' and y'' respectively. The three points z', x'', y'' are in the intersection of a local stable arc and a local unstable arc both contained in D and so the orbits of x'' and y'' α-shadow that of z and the orbit of x'' α-shadows the orbit of y'', contradicting 2-expansiveness. □

5.2.2 Bi-asymptotic sectors with spines

Let D be a bi-asymptotic sector with $\partial D = a^s \cup a^u$, where a^s is a stable arc and a^u is an unstable arc.

Definition 5.2.7. A non trivial continuum $C^s_D(p)$ ($C^u_D(p)$) is a *stable spine* (resp. *unstable spine*) if it does not separate D.

As before, we consider the map $g\colon a^u \to a^u$ defined by $a^u \cap C^s_D(x) = \{x, g(x)\}$. Recall that, by Lemma 5.2.3, g is continuous and reverses orientation. As a consequence if a point $p \in a^u$ is in a stable spine then p is a fixed point of g.

Lemma 5.2.8. *Bi-asymptotic sectors contain at most one stable spine and one unstable spine.*

Proof. Since g is a homeomorphism of an arc and it reverses orientation we have that g has exactly one fixed point. So there is at most one stable spine. Similarly there is at most one unstable spine. □

Lemma 5.2.9. *If $\Omega(f) = S$ and D is a bi-asymptotic sector then if there is a stable spine then there is an unstable one and it cuts the unstable spine in D.*

Proof. By contradiction suppose that there is a stable spine and an unstable spine and they are disjoint . Denote by S^s and S^u the stable and the unstable spines respectively. For all $x \in S^u$ we have that $C^s_D(x)$ separates D because if this were not the case the spines meets at x. We can define a partial order in S^u as $x < y$ if $C^s_D(y)$ separates x and a^s. It is easy to see that there is a minimal $z \in S^u$ with respect to this order. In this way we find a bi-asymptotic sector $D' \subset D$ bounded by a sub-arc of a^u and an arc in $C^s_D(z)$. Arguing in a similar way we find another bi-asymptotic sector $D'' \subset D'$ without spines. Then D'' is a regular sector, contradicting Proposition 5.2.6.

Now if there is a stable spine but no unstable one in a similar way we may find a regular bi-asymptotic sector leading again to a contradiction. □

Theorem 5.2.10. *If f is 2-expansive and $\Omega(f) = M$ then there are no bi-asymptotic sectors and consequently f is expansive.*

Proof. First notice that every regular stable leaf meets twice every unstable leaf. That is because there are exactly one stable spine and one unstable spine in D. Moreover, both cuts lie in different components of the complement in D of the union of the stable with the unstable leaves. Then there is a local product structure around every point in D away from the spines. Since we are assuming that there are not wandering points we conclude that periodic points are dense in D (arguing as for Anosov diffeomorphisms).

Take $p \in D$ a periodic point not in a spine. Denote by q the other point in the intersection of $C^s_D(p)$ with $C^u_s(D)$. Given $\delta > 0$ we can assume that $\text{dist}(f^n p, f^n q) \leq \delta$ for all $n \in \mathbb{Z}$. Since p is periodic we have k such that $f^k(p) = p$. Obviously, $f^{n+jk}(p) = f^n(p)$ for all $n, j \in \mathbb{Z}$. Therefore

$$\text{dist}(f^{n+jk}p, f^{n+jk}q) = \text{dist}(f^n p, f^n(f^{jk}q)) \leq \delta$$

for all $j, n \in \mathbb{Z}$. Then, the points p and $f^{jk}q$, with $j \in \mathbb{Z}$, contradicts the expansiveness of f for the expansive constant δ. Since δ is arbitrary we conclude that bi-asymptotic sectors can not exist if $\Omega(f) = M$ and f is 2-expansive.

Finally by Theorem 5.1.9 we have that f is expansive. □

5.3 An example

In this section we present an example of a surface 2-expansive homeomorphism with wandering points that is not expansive. [2] The construction of this example is based on the construction of a quasi-Anosov diffeomorphism given in [32].

Consider S_1 and S_2 two disjoint copies of the torus $\mathbb{R}^2/\mathbb{Z}^2$. Let $f_i : S_i \to S_i$ be two diffeomorphisms such that:

- f_1 is a derived-from-Anosov (see for example [111] Section 7.8 for a construction of such a map),
- f_2 is conjugated with f_1^{-1},
- f_i has a fixed point p_i, p_1 is a source and p_2 is a sink,

Also assume that there are local charts $\varphi_i : D \to S_i$, $D = \{x \in \mathbb{R}^2 : \|x\| \leq 2\}$, such that

1. $\varphi_i(0) = p_i$,
2. the pull-back of the stable (unstable) foliation by φ_1 (φ_2) is the vertical (horizontal) foliation on D and
3. $\varphi_1^{-1} \circ f_1^{-1} \circ \varphi_1(x) = \varphi_2^{-1} \circ f_2 \circ \varphi_2(x) = x/4$ for all $x \in D$.

Let A be the annulus $\{x \in \mathbb{R}^2 : 1/2 \leq \|x\| \leq 2\}$ and $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ the inversion $\psi(x) = x/\|x\|^2$. Consider $\hat{D}$ the open disk $\{x \in \mathbb{R}^2 : \|x\| < 1/2\}$. On $[S_1 \setminus \varphi_1(\hat{D})] \cup [S_1 \setminus \varphi_2(\hat{D})]$ consider the equivalence relation generated by

$$\varphi_1(x) \sim \varphi_2 \circ \psi(x)$$

for all $x \in A$. Denote by $\overline{x}$ the equivalence class of x. The surface

$$S = \frac{[S_1 \setminus \varphi_1(\hat{D})] \cup [S_1 \setminus \varphi_2(\hat{D})]}{\sim}$$

[2] This example was previously considered jointly with Joaquin Brum and Rafael Potrie, during a seminar course delivered by Jorge Lewowicz and José Vieitez.

is a bitorus with the quotient topology. The stable and unstable foliations are illustrated in Figure 5.6.

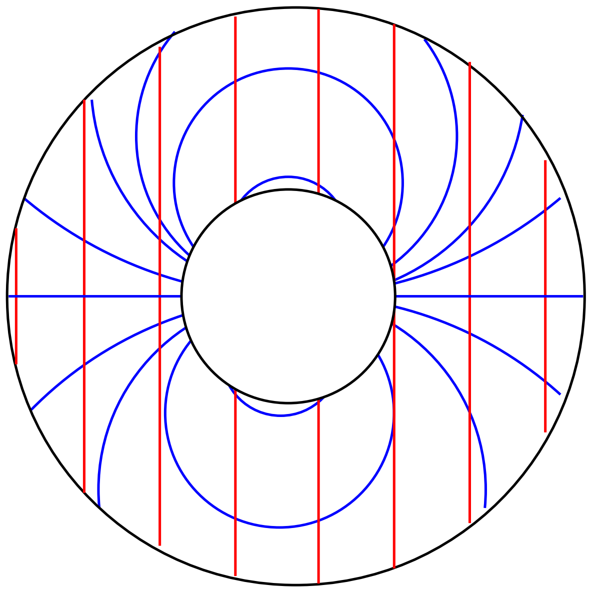

Figure 5.6: Foliations in the annulus $\overline{A}$. Blue lines represent the unstable foliation (after the inversion) and the red lines are the stable foliation.

Consider the homeomorphism $f\colon S \to S$ defined by

$$f(\overline{x}) = \begin{cases} \overline{f_1(x)} & \text{if } x \in S_1 \setminus \varphi_1(\hat{D}) \\ \overline{f_2(x)} & \text{if } x \in S_2 \setminus \varphi_2(D) \end{cases}$$

Theorem 5.3.1. *There are 2-expansive homeomorphisms of surfaces that are not expansive.*

Proof. We will show that f is 2-expansive but it is not expansive. It is not expansive because $\Omega f \neq S$.

To show that it is 2-expansive notice that

- Ωf is expansive (because it is hyperbolic) and
- Ωf is isolated, i.e. there is an open set U such that $\Omega f = \cap_{n\in\mathbb{Z}} f^n U$.

So, it only rest to show that there is $\delta > 0$ such that if $X \cap \Omega f = \emptyset$ and $\operatorname{diam} f^n X < \delta$ for all $n \in \mathbb{Z}$ then $|X| < 3$. Let $\overline{A} = \{\overline{x} : x \in \varphi_1(A)\}$. Let $\delta > 0$ be such that $B_\delta(x) \subset f^{-1}(\overline{A}) \cup \overline{A} \cup f(\overline{A})$ for all $x \in \overline{A}$. By construction, we have that $W^s_\delta(x) \cap W^u_\delta(x)$ has at most two points if $x \in \overline{A}$. Notice that for all $x \notin \Omega f$ there is $n \in \mathbb{Z}$ such that $f^n x \in \overline{A}$. This finishes the proof. □

5.4 Omega-expansivity

Let M be a smooth compact surface without boundary. Given a diffeomorphisms $f\colon M \to M$ define $\operatorname{Per}(f)$ as the set of periodic points of f and the non-wandering set $\Omega(f)$ as the set of those $x \in S$ satisfying: for all $\varepsilon > 0$ there is $n \geq 1$ such that $B_\varepsilon(x) \cap f^n(B_\varepsilon(x)) \neq \emptyset$. Recall that f satisfies Smale's *Axiom A* if $\text{clos}(\text{Per}(f)) = \Omega(f)$ and $\Omega(f)$ is hyperbolic. Recall that $\Lambda \subset M$ is *hyperbolic* if it is compact, invariant and the tangent bundle over Λ splits as $T_\Lambda M = E^s \oplus E^u$ the sum of two sub-bundles invariant by df and there are $c > 0$ and $\lambda \in (0, 1)$ such that:

1. if $v \in E^s$ then $\|df^n(v)\| \leq c\lambda^n\|v\|$ for all $n \geq 0$ and
2. if $v \in E^u$ then $\|df^n(v)\| \leq c\lambda^n\|v\|$ for all $n \leq 0$.

Definition 5.4.1. We say that f is Ω-*expansive* if $f\colon \Omega(f) \to \Omega(f)$ is expansive. We say that f is *robustly* Ω-*expansive* if there is an open C^1-neighborhood U of f such that every $g \in U$ is Ω-expansive.

Definition 5.4.2. A C^1 diffeomorphism $f\colon M \to M$ is Ω-stable if there is a C^1 neighborhood U of f such that for all $g \in U$ there is a homeomorphism $h\colon \Omega(f) \to \Omega(g)$ such that $h \circ f = g \circ h$.

Recall that f is a *star diffeomorphism* if there is a C^1 neighborhood U of f such that every periodic point of every $g \in U$ is hyperbolic. If f satisfies the axiom A then $\Omega(f)$ decomposes in a finite disjoint union basic sets $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_l$. A collection $\Lambda_{i_1}, \ldots, \Lambda_{i_k}$ is called a *cycle* if there exist points $a_j \notin \Omega(f)$, for $j = 1, \ldots, k$, such that $\alpha(a_j) \subset \Lambda_{i_j}$ and $\omega(a_j) \subset \Lambda_{i_{j+1}}$ (with $k + 1 \equiv 1$). We say that f *has not cycles* if there are not cycles among the basic sets of $\Omega(f)$. See for example [111] for the definition of basic set and more on this subject. Let us cite a well known result.

Theorem 5.4.3. *The following statements are equivalent:*

1. *f satisfies axiom A and has not cycles,*
2. *f is Ω-stable,*
3. *f is a star diffeomorphism.*

Proof. $(1 \Rightarrow 2)$. It was proved by Smale in 1970 [122].

$(2 \Rightarrow 3)$. It was proved by Franks in 1971 [31].

$(3 \Rightarrow 1)$. It was proved in 1992 by Aoki [2] and Hayashi [46]. □

We add another equivalence.

Theorem 5.4.4. *A C^1 diffeomorphism is robustly Ω-expansive if and only if it is Ω-stable.*

Proof. If f is Ω-stable then f satisfies Smale's axiom A. Therefore $\Omega(f)$ is hyperbolic and consequently $f\colon \Omega(f) \to \Omega(f)$ is expansive. Since f is Ω-stable we have that f is robustly Ω-expansive.

In order to prove the converse, suppose that f is robustly Ω-expansive. Using Franks' Lemma it is easy to see that f is a star diffeomorphism. □

5.5 Stable N-expansive surface diffeomorphisms

In this section we will consider a diffeomorphism f of a C^∞ compact surface S. As before, the *stable set* of $x \in S$ is

$$W^s_f(x) = \{y \in S : \lim_{n\to+\infty} \mathrm{dist}(f^n(x), f^n(y)) = 0\}.$$

The *unstable set* is defined by $W^u_f(x) = W^s_{f^{-1}}(x)$. Assume that f is Ω-stable, E^s, E^u are one-dimensional and define $I = [-1, 1]$. Let us recall the following fundamental result for future reference.

Theorem 5.5.1 (Stable manifold theorem)**.** *Let $\Lambda \subset S$ be a hyperbolic set of a C^r diffeomorphisms f of a compact surface S. Then, for all $x \in \Lambda$, $W^s_f(x)$ is an injectively immersed C^r submanifold. Also the map $x \mapsto W^s_f(x)$ is continuous: there is a continuous function $\Phi\colon \Lambda \to \mathrm{Emb}^r(I, S)$ such that for each $x \in \Lambda$ it holds that the image of $\Phi(x)$ is a neighborhood of x in $W^s_f(x)$. Finally, these stable manifolds also depend continuously on the diffeomorphisms f, in the sense that nearby diffeomorphisms yield nearby mappings Φ as above.*

Proof. See [99] Appendix 1. □

Definition 5.5.2. A C^r, $r \geq 1$, diffeomorphisms $f\colon S \to S$ is *Q^r-Anosov* if it is Ω-stable in the C^r topology and for all $x \in S$ there are $\delta_1, \delta_2 > 0$, a C^r coordinate chart $\varphi\colon U \subset S \to [-\delta_1, \delta_1] \times [\delta_2, \delta_2]$ such that $\varphi(x) = (0, 0)$ and two C^r functions $g^s, g^u\colon [-\delta_1, \delta_1] \to [\delta_2, \delta_2]$ such that the graph of g^s and g^u are the local expressions of the local stable and the local unstable manifold of x, respectively, and the degree r Taylor polynomials of g^s and g^u at 0 are different. If the polynomials coincide we say that there is an *r-tangency* at the intersection point.

Remark 5.5.3. *For $r = 1$ we have that Q^1-Anosov is quasi-Anosov, and in fact, given that S is two-dimensional, it is Anosov. For $r = 2$ we are requiring that if there is a tangency of a stable and an unstable manifold it is a quadratic one.*

Lemma 5.5.4. *If f is Ω-stable then for all $\varepsilon > 0$ there are $m \geq 0$ and a C^1 neighborhood $\mathcal{U}$ of f such that if $|n| \geq m$ then $g^n(x) \in B_\varepsilon(\Omega(g))$ for all $x \in M$ and $g \in \mathcal{U}$.*

Proof. By contradiction, take $\varepsilon > 0$, $g_k \to f$, $x_k \in M$ and $n_k \to \infty$ such that for all $k \in \mathbb{N}$, $\mathrm{dist}(g^i_k(x_k, \Omega(g_k)) \geq \varepsilon$ if $|i| \leq n_k$. By Theorem 8.3 in [120] and the Ω-stability of f we know that $\Omega(g_k) \to \Omega(f)$ in the Hausdorff metric. Therefore, if $x_k \to x$ then $\mathrm{dist}(f^i(x), \Omega(f)) \geq \varepsilon$ for all $i \in \mathbb{Z}$. But this is a contradiction because $\omega_f(x) \subset \Omega(f)$. □

Theorem 5.5.5. *In the C^r topology the set of Q^r-Anosov diffeomorphisms is an open set.*

Proof. We know that the set of Ω-stable diffeomorphisms is an open set. Let g_k be a sequence of Ω-stable C^r-diffeomorphisms converging to the C^r Ω-stable diffeomorphism f. Assume that g_k is not Q^r-Anosov for all $k \geq 0$. In order to finish the proof is it sufficient to show that f is not Q^r-Anosov. Since g_k is Ω-stable but it is not Q^r-Anosov, there is $x_k \in S$ with an r-tangency. Assume that $x_k \to x$. By Theorem 5.5.1 and Lemma 5.5.4 we have that there is an r-tangency at x. Therefore f is not Q^r-Anosov. Consequently, the set of Q^r-Anosov C^r-diffeomorphisms is an open set in the C^r topology. □

Definition 5.5.6. We say that a C^r diffeomorphism f is C^r*-robustly* N-expansive if there is a C^r neighborhood of f such that every diffeomorphism in this neighborhood is N-expansive.

Remark 5.5.7. *For $r = N = 1$ Mañé* [77] *proved that a diffeomorphism is robustly expansive if and only if it is quasi-Anosov.*

Lemma 5.5.8. *If $g \colon \mathbb{R} \to \mathbb{R}$ is a C^r functions with $r + 1$ roots in the interval $[a, b] \subset \mathbb{R}$ then $g^{(n)}$ has $r + 1 - n$ roots in $[a, b]$ for all $n = 1, 2, \ldots, r$ where $g^{(n)}$ stands for the n^{th} derivative of g.*

Proof. It follows by induction in n using the Rolle's theorem. □

Theorem 5.5.9. *Every Q^r-Anosov C^r diffeomorphism of a compact surface is C^r-robustly r-expansive.*

Proof. It follows by definitions and the previous Lemma. □

Corollary 5.5.10. *The example of Section 5.3 is a robustly 2-expansive C^2 diffeomorphism.*

Proof. By the construction, the diffeomorphism is C^∞. It is Ω-stable because the non-wandering set consists of a hyperbolic repeller and a hyperbolic attractor and there are no cycles. It only rests to note that the tangencies are quadratic because in local charts stable manifolds are straight lines and unstable manifolds are circle arcs. □

Chapter 6

Expansive flows

Let us start explaining the meaning of expansiveness of flows, from a kinematic viewpoint, discussing a well known physical example. Consider the differential equation of a simple pendulum: $\ddot{\theta} + \sin(\theta) = 0$. It is known since Galileo Galilei that the period of the oscillations is almost constant if the amplitude is small. But, if $T(\theta_0)$ is the period of an oscillation of amplitude θ_0 it can be proved that T is strictly increasing for $\theta_0 \in [0, \pi)$ (see for example [12] for a proof). Consider two close initial positions of the pendulum with vanishing initial velocities. Since the periods of the oscillations are different, we have that the solutions will be separated at some time. This is the meaning of kinematic expansiveness.

A key point for a pendulum clock as a practical timekeeper is that this separation time is large. In fact, it is easy to see that the separation is linear in time. This is a special feature of kinematic expansiveness, they are not so chaotic as a system with exponential error propagation.

If we consider the usual change of variables $x = \theta$ and $y = \dot{\theta}$, we can transform the equation of the pendulum into a first order differential equation in the plane. Consider two periodic solutions γ_1 and γ_2 bounding an annulus A in the plane. If $\phi\colon \mathbb{R} \times A \to A$ is the action of $\mathbb{R}$ on A induced by the pendulum equations we have our first example of a kinematic expansive flow on a compact surface. Precise definitions are given in Section 6.1.

The above considerations are related with the stability (or unstability) of trajectories in the sense of Lyapunov. In dynamical systems, another fundamental concept is the *structural stability* due to Andronov and Pontryagin. A system is structurally stable if there is a neighborhood of the system (in a specified topology) such that every system of this neighborhood has an *equivalent* behavior. In the discrete time case, two diffeomorphisms f, g of a manifold M are equivalent or *conjugated* if there is a homeomorphism $h\colon M \to M$ such that $f \circ h = h \circ g$. In the continuous time case, one can say that two flows ϕ and ψ are conjugated if there is a homeomorphism h as before such that $\phi_t \circ h = h \circ \psi_t$ for all $t \in \mathbb{R}$. This concept is very restrictive, because if there is a closed trajectory then its period should be preserved under perturbations. But this is impossible because slightly changing the velocities of the system one obtains a small perturbation (on any reasonable topology) and the periods of the perturbed system are different. Therefore, we must consider the concept of *topological equivalence*. Two flows are topologically equivalent if there is a homeomorphism that preserves trajectories and orientations. If the homeomorphism is the identity of the phase space we have that each flow is a (global) *time change* of the other. It is also called a *reparametrization* of the flow.

If one is allowed to change the velocities of single trajectories, the distance between two whole orbits can be measured with the *Fréchet distance*. If $\alpha, \beta \colon \mathbb{R} \to X$ are continuous curves on a metric space (X, dist), then the Fréchet or *geometric* distance between the curves is

$$\operatorname{dist}_F(\alpha, \beta) = \inf_h \sup_{t\in\mathbb{R}} \operatorname{dist}(\alpha(t), \beta(h(t))),$$

where $h \colon \mathbb{R} \to \mathbb{R}$ varies in the set of increasing homeomorphisms of $\mathbb{R}$. This takes us to the concept of *geometric expansiveness*, that is similar to kinematic expansiveness but allowing time reparametrizations of trajectories. This concept was first considered in the literature by Anosov to prove the structural stability of now called Anosov flows.

Later, Bowen and Walters introduced a definition of expansive flow, see [17], that on arbitrary compact metric spaces allowed them to prove some properties shared with Anosov flows. Their definition is of a *geometric* nature, that is, they require that trajectories are separated even allowing time changes of single orbits. In [17] it is noticed that kinematic expansiveness is not enough in order to recover results of hyperbolic flows. In the introduction of cited paper they consider a flow topologically equivalent with the pendulum system described above. Some of the results in [17] were generalized in [66] considering different families of reparametrizations and acting groups.

A different and very interesting kind of expansiveness was discovered by Gura. In [40], he proved that the horocycle flow of a surface with negative curvature is positive and negative (kinematic) separating, his definition requires to separate every pair of points in different orbits. He also proved a remarkable result: every *global* time change of such flow is positive and negative kinematic separating. It is known that horocycle flow is not geometric expansive.

The aim of this paper is to study kinematic expansiveness. Examples and basic properties are mainly stated on compact surfaces. A special feature of kinematic expansiveness is the non-invariance under global time changes. Therefore we also consider the definition of strong kinematic expansiveness requiring that every global time change must be kinematic expansive. A natural question is: why call it strong kinematic and not weak geometric? The answer can be found in the following example. Consider X a vector field in a two-dimensional torus $\mathbb{T}^2$ generating an irrational flow. Take a non-negative map ρ with just one zero at $p \in \mathbb{T}^2$. Define the vector field $Y = \rho X$ and let ϕ be its associated flow. As we will see, ϕ and its global time changes are kinematic expansive. The separation of trajectories is not geometric because generic orbits are parallel straight lines.

This chapter is organized as follows.

Section 6.1. We define and state the basic properties of expansive and separating flows in the kinematic, strong kinematic and geometric versions. Examples are given to analyze the relationship between the definitions.

Section 6.2. We consider flows on compact surfaces. We prove that on surfaces every geometric separating flow is geometric expansive (i.e. k^*-expansive in the sense of Komuro [69]). We also show that a flow on a surface is strong kinematic expansive if and only if it is strong separating; and also equivalent with: its singularities are of saddle type and the union of the separatrices is dense in the surface.

Section 6.3. We study the kinematic expansiveness of suspension flows. We found a dynamical characterization of the topology of compact subsets of the real line related with kinematic expansive suspensions. We give a characterization of arc homeomorphisms admitting a kinematic expansive suspension. We prove that the only C^1 diffeomorphism of an interval admitting a kinematic expansive suspension is the identity. A similar study is done for circle homeomorphisms and diffeomorphisms.

Section 6.4. We consider kinematic expansive flows of surfaces. We study the relationship between singularities and kinematic expansiveness in the disc and in the annulus. We show that every compact surface admits a kinematic expansive flow.

Section 6.5. We prove that hyper-expansive flows consist only in a finite number of singular points.

6.1 Hierarchy of expansive flows

In this section we present the main definitions. Let (X, dist) be a compact metric space and $\phi\colon \mathbb{R} \times X \to X$ be a continuous flow. We say that $p \in X$ is a *singularity* or an *equilibrium* point of ϕ if $\phi_t(p) = p$ for all $t \in \mathbb{R}$. To understand any definition of expansive flow one must consider the following simple fact.

Remark 6.1.1. *It holds that if there is at least one non-singular point $x \in X$ then for all $\delta > 0$ there exists $s \in \mathbb{R}$, such that $y = \phi_s(x) \neq x$ and for all $t \in \mathbb{R}$, $\operatorname{dist}(\phi_t(x), \phi_t(y)) < \delta$. Moreover, the value of s may be as small as we want.*

Then, if we are going to define a notion of expansive or separating flow we must take care of points in the same orbit. In the subject of expansive flows we consider the hierarchy shown in Table 6.1. The terms *kinematic* and *geometric* first appear in the literature of expansive systems (to our best knowledge) in [24] (page 138). Definitions in the left column of the table separate every pair of points not being in the same *local* orbit, and the ones in the right separate points in

Geometric Expansive	$\Rightarrow$	Geometric Separating
$\Downarrow$		$\Downarrow$
Strong Kinematic Expansive	$\Rightarrow$	Strong Separating
$\Downarrow$		$\Downarrow$
Kinematic expansive	$\Rightarrow$	Separating

Table 6.1: Hierarchy of expansive flows

different *global* orbits. Strong expansiveness deals with time changes of the whole flow and the geometric notions allows time changes of single orbits. As the reader will see, the implications indicated by the arrows are easy to prove. Now we give the precise definitions, examples and counterexamples showing that no arrow in the table can be reversed in the general setting of compact metric spaces. We also state some basic properties.

6.1.1 Kinematic expansive flows

Let us start with the main notion of the present chapter.

Definition 6.1.2. We say that ϕ is *kinematic expansive* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\operatorname{dist}(\phi_t(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ then there exists $s \in \mathbb{R}$ such that $|s| < \varepsilon$ and $y = \phi_s(x)$.

In [66] kinematic expansiveness is considered with the name $\{\mathrm{id}\}$-*expansiveness*. This means that the only reparametrization allowed is the identity of $\mathbb{R}$. This definition was also mentioned in (the first section of) [17].

Two continuous flows $\phi\colon \mathbb{R} \times X \to X$ and $\psi\colon \mathbb{R} \times Y \to Y$ are said to be *equivalent* if there exists a homeomorphism $h\colon X \to Y$ such that $\phi_t = h^{-1} \circ \psi_t \circ h$ for all $t \in \mathbb{R}$.

Remark 6.1.3. *Clearly, kinematic expansiveness is invariant under flow equivalence, i.e., it does not depend on the metric defining the topology of X.*

A continuous flow $\psi\colon \mathbb{R} \times X \to X$ is *topologically equivalent* with $\phi\colon \mathbb{R} \times Y \to Y$ if there is a homeomorphism $h\colon X \to Y$ such that for each $x \in X$ the orbits $\phi_{\mathbb{R}}(h(x))$ and $\psi_{\mathbb{R}}(x)$ and its orientations coincide. If in addition, the homeomorphism h is the identity of X, we say that ϕ is a *time change* of ψ.

The following example is topologically equivalent with the pendulum system (restricted to an annulus as mentioned above) and shows that a time change can destroy kinematic expansiveness.

Example 6.1.4 (Periodic band)**.** *Consider the annulus in the plane*

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \in [1, 4]\}$$

bounded by circles of radius 1 and 2. A flow ϕ on A can be defined by the equation

$$(\dot{x}, \dot{y}) = \frac{1}{\sqrt{x^2 + y^2}}(-y, x).$$

The solutions are circles as shown in Figure 6.1. It is easy to see that this flow is kinematic expansive (there is a proof in Example 2 of [66] *page 84). The equation $(\dot{x}, \dot{y}) = (-y, x)$ defines*

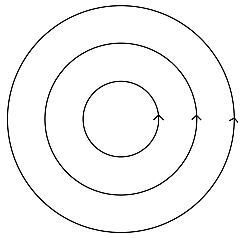

Figure 6.1: Periodic orbits in the annulus.

a time change of ϕ that is not kinematic expansive because the angular velocities are constant.

Remark 6.1.5. *At the end of Definition 6.1.2 above, we required that the points x and y are in a orbit segment of small time. Since our phase space is compact, this segment has a small diameter too. Notice that the converse is true if there are no singularities, i.e., orbit segments of small diameter are defined by small times. But if there is a singularity accumulated by regular orbits this is no longer true as the reader can verify, just apply the continuity of the flow at the singular point.*

In spite of this remark we will show that if we require in Definition 6.1.2 that x and y are in a orbit segment of small diameter (instead of small time) we obtain an equivalent definition. Let us first introduce another distance in X by

$$\text{dist}_\phi(x, y) = \inf\{\text{diam}(\phi_{[a,b]}(z)) : z \in X, [a, b] \subset \mathbb{R}, x, y \in \phi_{[a,b]}(z)\}$$

if x, y are in the same orbit and $\text{dist}_\phi(x, y) = \infty$ in other case. Of course, this metric will define a different topology on X if X is not just a periodic orbit.

Proposition 6.1.6. *A flow ϕ is kinematic expansive if and only if for all $\beta > 0$ there exists $\delta > 0$ such that if $\text{dist}(\phi_t(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ then $\text{dist}_\phi(x, y) < \beta$.*

Proof. ($\Rightarrow$) Consider $\beta > 0$ and take $\varepsilon > 0$ such that $\text{dist}(\phi_t(x), x) \leq \beta$ for all $x \in X$ and $t \in [-\varepsilon, \varepsilon]$. By hypothesis, there exists δ such that if $\text{dist}(\phi_t(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ then there is $s \in \mathbb{R}$ such that $|s| < \varepsilon$ and $\phi_s(x) = y$. Then $\text{dist}_\phi(x, y) < \beta$ and the proof ends.

($\Leftarrow$) First we fix $\varepsilon > 0$. Take any $\beta_1 > 0$ and by hypothesis there exists δ_1 such that if $\text{dist}(\phi_t(x), \phi_t(y)) < \delta_1$ for all $t \in \mathbb{R}$ then $\text{dist}_\phi(x, y) < \beta_1$. It is easy to see that there is just a finite number of orbits with diameter smaller than $\delta_1/2$. Now take $\beta_2 > 0$ such that if $\mathrm{diam}(\phi_{\mathbb{R}}(p)) < \beta_2$ then p is a singular point. For this value of β_2 there is an expansive constant δ_2 (by hypothesis).

Take $\rho < \delta_2/2$ and denote by Sing the set of singular points of the flow. It is easy to see that for all $x \in B_\rho(\mathrm{Sing}) = \cup_{q\in \mathrm{Sing}} B_\rho(q)$, $x \notin \mathrm{Sing}$, there exists $t_0 \in \mathbb{R}$ such that $\phi_{t_0}(x) \notin B_\rho(\mathrm{Sing})$. We will prove that there is $\beta_3 \in (0, \beta_2)$ such that if $x \notin B_\rho(\mathrm{Sing})$ and $\mathrm{diam}(\phi_{[0,t]}(x)) < \beta_3$ then $|t| < \varepsilon$. By contradiction suppose there exists $x_n \to z$, $x_n \notin B_\rho(\mathrm{Sing})$ and $t_n \to \infty$ such that $\mathrm{diam}(\phi_{[0,t_n]}(x_n)) \to 0$. This implies that $z \in \mathrm{Sing}$ which is a contradiction.

Finally we claim that δ_3 is an expansive constant associated to ε. In order to prove it, suppose that $\mathrm{dist}(\phi_t(x), \phi_t(y)) < \delta_3$ for all $t \in \mathbb{R}$. We can assume that x is not a singular point and then there exists $t_0 \in \mathbb{R}$ such that $\phi_{t_0}(x) \notin B_\rho(\mathrm{Sing})$. Then

$$\mathrm{dist}(\phi_t(\phi_{t_0}(x)), \phi_t(\phi_{t_0}(y))) < \delta_3$$

and the hypothesis implies that there is $s \in \mathbb{R}$ such that $\phi_s(\phi_{t_0}(x)) = \phi_{t_0}(y)$ and also the diameter of $\phi_{[0,s]}(\phi_{t_0}(x))$ is smaller than β_3. Since $\phi_{t_0} \notin B_\rho(\mathrm{Sing})$ we have that $|s| < \varepsilon$ and then $\phi_s(x) = y$ with $|s| < \varepsilon$. □

6.1.2 Strong kinematic expansive flows

As we saw in Example 6.1.4 kinematic expansiveness is not an invariant property under time changes of flows. Therefore the following definition is natural.

Definition 6.1.7. A flow is said to be *strong kinematic expansive* if every time change is kinematic expansive.

Example 6.1.8. *Consider an irrational flow (every orbit is dense) on the two dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with velocity field X. Take any non-negative smooth function f with just one zero at some point p in the torus. Denote by ϕ the flow generated by the vector field fX. Such flow is illustrated in Figure 6.2. To show that ϕ is strong kinematic expansive consider any time change of the flow. The idea is the following. Take two points being in different local orbits and wait until one of them is very close to p. By continuity this point will stay close to p for a long time while the other point will be separated. This argument will be formalized later in Theorem 6.2.8.*

Remark 6.1.9. *The periodic flow in the annulus shown in Example 6.1.4 is kinematic expansive but not in the strong sense.*

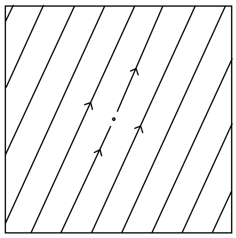

Figure 6.2: Irrational flow in the torus with a fake saddle.

6.1.3 Geometric expansive flows

The idea of geometric expansiveness is that the trajectories separate even if one allows a time change of the trajectories. Denote by $\mathcal{H}^+$ the set of all increasing homeomorphisms $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$. Such homeomorphisms will be called *time reparametrizations.*

Definition 6.1.10. We say that ϕ is *geometric expansive* if for all $\beta > 0$ there exists $\delta > 0$ such that if $\operatorname{dist}(\phi_{h(t)}(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ with $h \in \mathcal{H}^+$ then x, y are in a orbit segment of diameter smaller than β.

In the literature these flows are simply called *expansive*. In the case of regular flows, i.e. without equilibrium points, it is equivalent with the one given by R. Bowen and P. Walters in [17]. For the general case (i.e. with or without singular points) the definition is equivalent with the given by M. Komuro in [69] (see [5] for a proof). Examples of geometric expansive flows are suspensions of expansive homeomorphisms [17], Anosov flows, the Lorenz attractor [69] and singular suspensions of expansive interval exchange maps [5].

Remark 6.1.11. *It is easy to see that Examples 6.1.4 and 6.1.8 are not geometric expansive.*

6.1.4 Separating flows

The term *separating* was first used in [39, 40]. This kind of expansiveness only separates points in different global orbits.

Definition 6.1.12. A flow ϕ is *separating* if there is $\delta > 0$ such that if $\operatorname{dist}(\phi_t(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ then $y \in \phi_{\mathbb{R}}(x)$.

Example 6.1.13 (Minimal separating flow in the torus)**.** *In* [29] *it is defined a continuous (non-smooth) time change of an irrational flow on the two dimensional torus T^2 with the following property: the set $\{(\phi_t(x), \phi_t(y)) : t \geq 0\}$ is dense in $T^2 \times T^2$ whenever x and y are on different orbits in T^2. Clearly, it implies that the flow is separating. We were not able to decide if this example is kinematic expansive or not.*

The following is an easy example showing that there are separating flows that are not kinematic expansive.

Example 6.1.14 (A separating flow in the Möebius Band)**.** *Consider the map* $f\colon[-1,1]\to[-1,1]$ *given by* $f(x)=-x$ *and consider* $T(x)=1+x^2$ *for all* $x\in[-1,1]$*. It is easy to see that the suspension flow of* f *with return time* T *is a separating flow in the Möebius band. See Figure 6.3.*

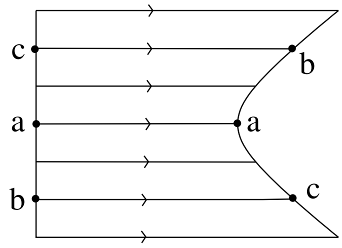

Figure 6.3: A separating flow in the Moebius band.

Remark 6.1.15. *The previous example in the Moebius band is not kinematic expansive. Consider two points as c and b in Figure 6.3. They are in the same orbit but not in a small orbit segment. Taking them close to the point a we contradict kinematic expansiveness.*

Let us give some general remarks that hold for every notion of expansiveness considered in this article. Recall that the definition of *separating flow* is the weaker in Table 6.1.

Definition 6.1.16. A singularity p of ϕ is *ϕ-isolated* if there is $\delta>0$ such that for all $x\in B_\delta(p)$, $x\neq p$, there is $t\in\mathbb{R}$ such that $\phi_t(x)\notin B_\delta(p)$.

Remark 6.1.17. *If* ϕ *is separating and* $p\in X$ *is a singular point of the flow then* p *is* ϕ*-isolated. In particular, the set of singular points is finite.*

6.1.5 Strong separating flows

Definition 6.1.18. A flow is *strong separating* if every time change is separating.

The following is a remarkable example.

Example 6.1.19. *In* [40] *it is shown that the horocycle flow on a surface of negative curvature is strong separating. In fact, Gura shows that the separation of trajectories occurs in positive and in negative times.*

Remark 6.1.20. *The example shown in* [29] *(recall Example 6.1.13) is separating but it is not strong separating.*

6.1.6 Geometric separating flows

Definition 6.1.21. A flow ϕ is said to be *geometric separating* if there exists $\delta > 0$ such that if $\mathrm{dist}(\phi_{h(t)}(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ and some $h \in \mathcal{H}^+$ then $y \in \phi_{\mathbb{R}}(x)$.

To study geometric separating flows we introduce a natural definition for dynamics with discrete time.

Definition 6.1.22. We say that a homeomorphism $f\colon Y \to Y$ is *separating*[1] if there is $\delta > 0$ such that if $\mathrm{dist}(f^n(x), f^n(y)) < \delta$ for all $n \in \mathbb{Z}$ then there is $m \in \mathbb{Z}$ such that $y = f^m(x)$.

Proposition 6.1.23. *A suspension is geometric separating if and only if the suspended homeomorphism is separating.*

Proof. Is similar to the proof of Theorem 6 in [17]. □

Now we give an example showing that separating homeomorphisms may not be expansive.

Example 6.1.24. *Let X be the subset of the sphere $\mathbb{R}^2 \cup \{\infty\}$ given by*

$$X = \{\infty\} \cup \{(n, 0) : n \in \mathbb{Z}\} \cup \{(n, \pm 1/m) : n \in \mathbb{Z}, m \in \mathbb{Z}^+, |n| \le m\}.$$

Define $f\colon X \to X$ as $f(\infty) = \infty$, $f(n, 0) = (n + 1, 0)$, $f(n, \pm 1/m) = (n + 1, \pm 1/m)$ if $n < m$ and $f(m, \pm 1/m) = (-m, \mp 1/m)$. It is easy to see that f is a homeomorphism. It is not expansive because the points $(0, 1/m)$ and $(0, -1/m)$ contradicts expansiveness for arbitrary small expansive constants. It is a separating homeomorphism because these are the only points contradicting expansiveness and they are in the same orbit. Therefore, the suspension of this example is not geometric expansive but it is geometric separating.

Remark 6.1.25. *We also have that the suspension flow in the previous example is strong separating but it is not strong kinematic expansive.*

6.1.7 Summary of counterexamples

We have defined six variations of expansive and separating flows on compact metric spaces. In the following table we recall the counterexamples in the hierarchy:

As we can see in the diagram, the six definitions are different in the general context of continuous flows on compact metric spaces.

[1]In [39] Gura calls *separating* to what we call *expansive homeomorphism*. We use the expression *separating homeomorphism* with a different meaning.

Geometric Expansive	$\not\Leftarrow$ Example 6.1.24	Geometric Separating
$\not\Uparrow$ Examples 6.1.8 and 6.1.19		$\not\Uparrow$ Examples 6.1.8 and 6.1.19
Strong Kinematic Expansive	$\not\Leftarrow$ Example 6.1.24	Strong Separating
$\not\Uparrow$ Example 6.1.4		$\not\Uparrow$ Example 6.1.13
Kinematic expansive	$\not\Leftarrow$ Example 6.1.14	Separating

Table 6.2: Diagram of counterexamples

6.2 Hierarchy of expansiveness on surfaces

In this section we will show that the hierarchy of expansive flows presented in Table 6.1 is simpler, see Table 6.2, if we assume that the phase space is a compact surface. The first

$$\boxed{\text{Geometric Expansive} \Leftrightarrow \text{Geometric Separating}}$$
$$\Downarrow$$
$$\boxed{\text{Strong Kinematic Expansive} \Leftrightarrow \text{Strong Separating}}$$
$$\Downarrow$$
$$\boxed{\text{Kinematic expansive}}$$
$$\Downarrow$$
$$\boxed{\text{Separating}}$$

Table 6.3: Hierarchy of expansive flows of compact surfaces

equivalence in Table 6.3 is given in Theorem 6.2.7 and the second one is proved in Theorem 6.2.8. To prove these results we will first study the local behavior near singular points and time changes of flows with wandering points.

6.2.1 Isolated singular points

In this section we study the local behavior of singularities of separating flows of surfaces. Let $\phi\colon \mathbb{R} \times S \to S$ be a continuous flow on a compact surface S. As mentioned in Remark 6.1.17, every singular point is ϕ-isolated if ϕ is separating.

Let us introduce some definitions. A regular orbit γ is a *separatrix* of $p \in \mathrm{Sing}$ if for $x \in \gamma$ it holds that $\lim_{t\to+\infty} \phi_t(x) = p$ (*unstable separatrix*) or $\lim_{t\to-\infty} \phi_t(x) = p$ (*stable separatrix*). A singular point is said to be a *(multiple) saddle* if it presents a finite number of separatrices. We say that $p \in \mathrm{Sing}$ is an *n-saddle* if p is a multiple saddle of index $1 - n$ (i.e. if it has $n+1$

stable separatrices).

Recall that a singular point p is (Lyapunov) *stable* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\text{dist}(x, p) < \delta$ then $\text{dist}(\phi_t(x), p) < \varepsilon$ for all $t \geq 0$. We say that p is *asymptotically stable* if it is stable and there is $\delta_0 > 0$ such that if $\text{dist}(x, p) < \delta_0$ then $\phi_t(x) \to p$ as $t \to +\infty$. If p is asymptotically stable we say that p is a *sink*. We say that p is a *source* if it is a sink for ϕ^{-1} defined as $\phi_t^{-1} = \phi_{-t}$.

Let us recall from [44] some well known facts and notations relative to the Poincaré-Bendixon Theory. Let $p \in \text{Sing}$ be a ϕ-isolated singular point. Consider a Jordan curve C bounding a neighborhood U of p such that if $\phi_{\mathbb{R}}(x) \subset \operatorname{clos} U$ then $x = p$. If for some $y \in C$ it holds that $\phi_{\mathbb{R}^+}(y) \subset U$ then we say that $\phi_{\mathbb{R}^+}(y)$ is a *stable separatrix arc* (or a *base solution* in the terminology of [44]). Since p is ϕ-isolated, we have that $\lim_{t\to\infty} \phi_t(y) = p$. In the same conditions, if $\phi_{\mathbb{R}^-}(y) \subset U$ then this orbit segment is called *unstable separatrix arc*.

Suppose that $y_1, y_2 \in C$ determine two separatrix arcs. An open subset S bounded by p, the separatrix arcs of y_1 and y_2, and an arc in C from y_1 to y_2 is called a *sector*. Notice that each pair of separatrix arcs determines two sectors.

A sector σ is *hyperbolic* if contains no separatrix arc. A sector σ determined by two stable (or two unstable) separatrix arcs is called *parabolic* if it contains no unstable (or stable) separatrix arc. With reference to [44], elliptic sectors needs not to be consider because p is ϕ-isolated. The number of hyperbolic sectors is finite by Lemma 8.2 in [44].

Proposition 6.2.1. *Assume that U is an isolating neighborhood of $p \in \text{Sing}$ bounded by a Jordan curve C as above. If the closures of all the hyperbolic sectors are deleted from U then the remaining set is either:*

1. *empty and p is a multiple saddle,*
2. *U and p is a sink or a source or*
3. *the union of a finite number of pairwise disjoint parabolic sectors.*

Proof. See Lemma 8.3 of [44]. □

In Figure 6.4 the three possible cases of Proposition 6.2.1 are illustrated.

Definition 6.2.2. Let R be an embedded disc in S and define a rectangle $K = [-1, 1] \times [0, 1] \subset \mathbb{R}^2$. We say that:

1. R is a *regular flow box* if ϕ restricted to R is topologically equivalent with the constant vector field $X(x, y) = (1, 0)$ restricted to K,
2. R is a *parabolic flow box* if ϕ restricted to R is topologically equivalent with $X(x, y) = \pm(x, y)$ restricted to K,
3. R is a *hyperbolic flow box* if ϕ restricted to R is topologically equivalent with $X(x, y) = (x^2 + y^2, 0)$ restricted to K.

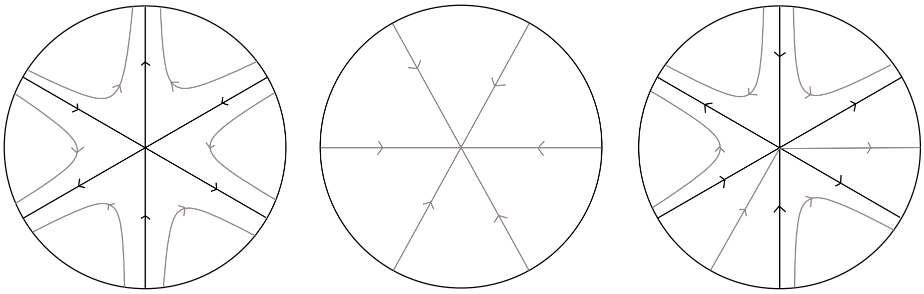

Figure 6.4: Examples of isolated singularities. Left: a multiple saddle. Center: a sink. Right: a combination of hyperbolic and parabolic sectors.

In the last two cases we say that R is a *singular flow box*.

Proposition 6.2.3. *If a flow ϕ on a compact surfaces S presents a finite number of isolated singularities then $S = \cup_{i=1}^n R_i$ where:*

- *each R_i is a regular or singular flow box and*
- *if $i \neq j$ then $R_i \cap R_j \subset \partial R_i \cap \partial R_j$.*

Proof. It follows by Proposition 4.3 of [41] and Proposition 6.2.1 above. □

6.2.2 Time changes and wandering points

Let ϕ be a continuous flow on a compact surface S.

Theorem 6.2.4. *If ϕ is a continuous flow on a compact surface S and ϕ has wandering points then there is a time change of ϕ that is not separating.*

Proof. If ϕ has a non-isolated singular point then ϕ is not separating. Therefore, we will assume that all the singularities are isolated.

Let $p \in S$ be a wandering point for ϕ. Then there is a compact arc l transverse to the flow having p in its interior such that $\phi_t(l) \cap l = \emptyset$ for all $t \neq 0$. Let $L = \phi_{\mathbb{R}}(l)$. Consider a covering of boxes $R_1, \dots, R_n$, $S = \cup_{i=1}^n R_i$, as in Proposition 6.2.3. Divide l with two interior points in three sub-arcs $l = l_1 \cup l_2 \cup l_3$ in such a way that $\phi_{\mathbb{R}}(l_2)$ intersects each ∂R_i only at the transversal part. It is possible because there is a finite number of flow segments in the boundary of the boxes R_i and $\phi_t(l) \cap l = \emptyset$ for all $t \neq 0$. We will show that there is a time change ψ of ϕ such that for all $\delta > 0$ there are $x, y \in l_2$, $x \neq y$, such that $\operatorname{dist}(\psi_t(x), \psi_t(y)) < \delta$ for all $t \in \mathbb{R}$.

Fix a box R_i such that $\phi_{\mathbb{R}}(l_2) \cap R_i \neq \emptyset$. Assume first that R_i is a regular flow box. The boundary of R_i is the union of two transversal arcs a and b and two orbit segments. Suppose that the flow enters to the box through a. Given two points $x, y \in a$ the sub-arc of a with

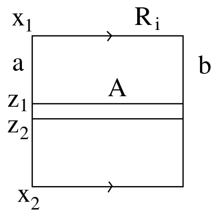

Figure 6.5:

extreme points x, y will be denoted by $[x, y]$. Call x_1 and x_2 the extreme points of a as shown in Figure 6.5.

Take $z_1, z_2 \in a$ such that $[z_1, z_2] \cap \phi_{\mathbb{R}}(l_2) = \emptyset$. Since R_i is a regular flow box there is a homeomorphism $h\colon R_i \to K = [-1,1] \times [0,1]$ taking orbit segments in R_i into horizontal segments in K. For each $p \in R_i$ denote by $\gamma(p)$ the preimage by h of the vertical segment through $h(p)$. Each $\gamma(p)$ is a compact arc transversal to the flow. Consider a time change ψ such that:

1. if $x \in [x_1, z_1]$ then $\psi_t(x) \in \gamma(\phi_t(x_1))$ for all $t \in [0, T_1]$ where $\phi_{T_1}(x_1) \in b$ and $\phi_{[0,T_1]}(x_1) \subset R_i$,
2. if $x \in [z_2, x_2]$ then $\psi_t(x) \in \gamma(\phi_t(x_2))$ for all $t \in [0, T_2]$ where $\phi_{T_2}(x_2) \in b$ and $\phi_{[0,T_2]}(x_2) \subset R_i$.

Now consider a hyperbolic box R_i. Again denote by $a = [x_1, x_2] \subset \partial R_i$ the transversal part of the boundary of R_i where the flow enters to the box. Consider $u_i, v_i \in a$, for $i \in \mathbb{Z}^+$, such that $u_1 < v_1 < u_2 < v_2 < \dots$ and $[u_i, v_i] \cap \phi_{\mathbb{R}}(l_2) = \emptyset$ for all $i = 1, 2, 3, \dots$. Denote by $p \in \partial R_i$ the singular point in the boundary of R_i. Again, with a homeomorphism $h\colon R_i \to K$ we have a transversal (vertical) foliation on $R_i \setminus \{p\}$. Inside R_i consider three flow boxes A_1, B_1, C_1 bounded by orbit segments and vertical arcs as in Figure 6.6. Also consider the hyperbolic flow

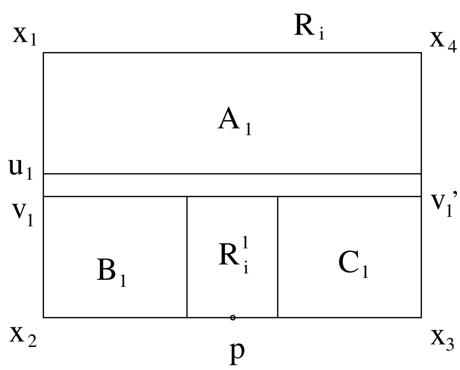

Figure 6.6:

box $R_i^1 \subset R_i$ as in the figure. Define

$$T_a = \sup\{t > 0 : \phi_{[0,t]}(x_1) \subset R_i\},$$
$$T_b = \sup\{t > 0 : \phi_{[0,t]}(x_2) \subset B_1\},$$
$$T_c = \sup\{-t > 0 : \phi_{[t,0]}(x_3) \subset C_1\},$$

where x_3 is the vertex of the box R_i shown in Figure 6.6. Consider the time change ψ satisfying:

1. if $x \in [x_1, u_1]$ then $\psi_t(x) \in \gamma(\phi_t(x_1))$ for all $t \in [0, T_a]$,
2. if $x \in [v_1, x_2]$ then $\psi_t(x) \in \gamma(\phi_t(x_2))$ for all $t \in [0, T_b]$,
3. if $x \in [v_1', x_3]$ then $\psi_{-t}(x) \in \gamma(\phi_{-t}(x_3))$ for all $t \in [0, T_c]$.

Inside R_i^1 consider a similar subdivision considering the orbit segments of u_2, v_2 as in Figure 6.7. Inductively we have a sequence of regular boxes A_k, B_k, R_k and hyperbolic boxes R_k^i.

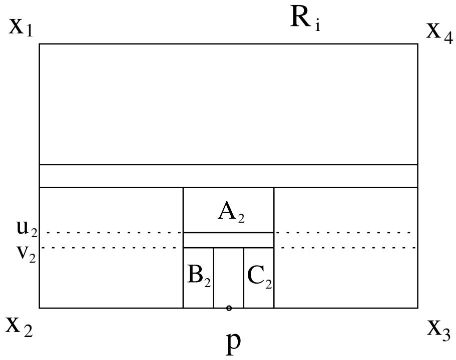

Figure 6.7:

On each R_k^i assume that ψ satisfies the corresponding conditions as in R_1^i. Assume that $\mathrm{diam}(R_i^k) \to 0$ as $k \to \infty$.

On parabolic boxes, assume that ψ coincides with ϕ.

In this way we obtain a (global) flow ψ that is a time change of ϕ and ψ is not separating because on each box $R_i \cap \phi_{\mathbb{R}}(l_2)$ the flow ψ preserves the vertical foliation of the box. □

6.2.3 Geometric separating and geometric expansive flows on surfaces

Let us recall that in [5] (see Theorem 6.7) it is proved that a flow on a compact surface S that is not a torus, is geometric expansive [2] if and only if the set of singular points is finite and there are neither wandering points nor periodic orbits. We do not consider singular points as periodic orbits.

Lemma 6.2.5. *If ϕ is a strong separating flow on a compact surface then ϕ has no periodic orbits.*

[2]Notice that in cited paper *expansive* means *geometric expansive* in the present terminology.

Proof. By Theorem 6.2.4 we have that there are no wandering points. Therefore, if γ is a periodic orbit, every point close to γ has to be periodic (this can be easily proved by considering a local cross section through γ and its first return map). But if a periodic orbit is accumulated by periodic orbits then there is a time change of ϕ that is not strong separating. Therefore, a strong separating flow cannot have periodic orbits. □

Lemma 6.2.6. *The torus does not admit geometric separating flows.*

Proof. Assume by contradiction that ϕ is a geometric separating flow on the torus. We know by Theorem 6.2.4 and Lemma 6.2.5 that ϕ has neither wandering points nor periodic orbits. Since ϕ is separating, we have that the singular points are ϕ-isolated. Applying Proposition 6.2.1 and the fact that there are no wandering points we have that every singular point is of saddle type, that is because there are neither sources, sinks nor parabolic sectors. Since the Euler characteristic of the torus equals zero we have that singular points are 0-saddles (sometimes called *fake saddles*). Consider another flow ψ that removes the singularities of ϕ, i.e., satisfying: 1) ψ has no singular points and 2) every orbit of ϕ is contained in a orbit of ψ. It is known that under these conditions (see for example Lemma 4.1 in [5]) ψ is an irrational flow, i.e. a suspension of an irrational rotation of the circle. But now it is easy to see that ϕ cannot be geometric separating. This contradiction proves the lemma. □

Theorem 6.2.7. *A continuous flow on a compact surface is geometric separating if and only if it is geometric expansive.*

Proof. We only have to prove the direct part because the converse holds on arbitrary compact metric spaces. Therefore, consider a geometric separating flow ϕ. By Theorem 6.2.4 we have that ϕ has no wandering points. By Lemma 6.2.6 we know that S is not the torus and by Lemma 6.2.5 we have that ϕ has no periodic orbits. Now, recalling that the set of singular points is finite we apply Theorem 6.7 in [5] to conclude that ϕ is geometric expansive. □

6.2.4 Strong kinematic expansive and strong separating flows on surfaces

In this section we prove the second equivalence of Table 6.3.

Theorem 6.2.8. *Let S be a compact surface and let ϕ be a continuous flow on S. The following statements are equivalent:*

1. *ϕ is strong kinematic expansive,*
2. *ϕ is strong separating,*
3. *the singular points are saddles and the union of their separatrices is dense in S.*

Proof. $(1 \to 2)$. It holds in the general setting of compact metric spaces.

$(2 \to 3)$. By Theorem 6.2.4 we have that ϕ has no wandering points. Therefore there are no parabolic sectors and singularities are of saddle type. By Lemma 6.2.5 we have that strong separating flows have no periodic orbits. So, as in Proposition 4 of [5] we conclude that the union of the separatrices is dense in S given that ϕ is strong separating. This proves that (2) implies (3).

$(3 \to 1)$. By Theorems 6.1 and 5.3 in [5] we only have to consider the case where S is a torus and the flow is minimal with a finite number of 0-saddles. Let γ be a global transversal to the flow. Denote by T the return time function of γ that is defined in $\gamma \setminus A$ where A is a finite set. In the points of A the map T diverges. Now take $u \neq v$ in $\gamma \setminus A$. Let $f \colon \gamma \to \gamma$ be the extended return map (notice that the points in A does not return to γ but this map can be extended by continuity to a minimal rotation f). Since A is finite, there is $\delta > 0$ such that if $\text{dist}(u, v) < \delta$, $a \in A$ and $f^{n_k}(u) \to a$ then $f^{n_k}(v) \to b$ with $b \notin A$. This implies that the flow will separate u and v (see the techniques of Proposition 6.3.1 below). □

Remark 6.2.9. *The only surface admitting a strong kinematic expansive flow that is not geometric expansive is the torus. Therefore, applying Theorem 6.5 in* [5]*, we have that a compact surface admits a strong kinematic expansive flow if and only if it is obtained from the torus attaching $h \geq 0$ handles, $b \geq 0$ boundaries and $c \geq 0$ cross-cups.*

Remark 6.2.10. *Every strong kinematic expansive flow of a compact surface is topologically equivalent with a C^∞ flow. This can be proved with Gutierrez's smoothing results as done in* [5] *for the geometric expansive case.*

Definition 6.2.11. If ϕ is a strong kinematic expansive flow we say that the expansive constant is *uniform* if for all $\beta > 0$ there is $\delta > 0$ such that if ψ is a time change of ϕ and $\text{dist}(\psi_t(x), \psi_t(y)) < \delta$ for all $t \in \mathbb{R}$ then x, y are in a orbit segment of diameter smaller than β.

Uniformity of the expansive constant means that there is an expansive constant working for every time change.

Remark 6.2.12. *From the arguments above we have that on surfaces every strong kinematic expansive flow has a uniform expansive constant.*

6.3 Suspension flows

Let ϕ be a continuous flow without singularities defined on a compact metric space X. We say that a compact subset $l \subset X$ is a *local section* around x if $x \in l$, there is $\tau > 0$ such that $l \cap \phi_{[-\tau,\tau]}(y) = \{y\}$ for all $y \in l$ and x is an interior point of $\phi_{(-\tau,\tau)}(l)$. A compact subset $l \subset X$ is a *global section* for ϕ if for all $x \in l$ there is a neighborhood U of x such that $U \cap l$ is a local cross section in the sense of Whitney [135] (see also [17]) and every orbit cuts l. If ϕ admits a global section $l \subset X$ we can consider the first return map $f\colon l \to l$ satisfying $f(x) = \phi_t(x)$ if $t > 0$ and $\phi_{(0,t]}(x) \cap l = \{f(x)\}$ for all $x \in l$. In this case we say that ϕ is a *suspension* of f.

6.3.1 Kinematic expansive suspensions

The expansiveness of a homeomorphism is known to be equivalent with the geometric expansiveness of each suspension (see [17]) and also to the kinematic expansiveness of a suspension of constant time (see [66]).

Here we consider the kinematic expansiveness of a suspension with arbitrary (continuous) return time.

Proposition 6.3.1. *Suppose ϕ is a suspension of $f\colon l \to l$ and let $T_k\colon l \to \mathbb{R}$ be such that for all $k \in \mathbb{Z}$ and $x \in l$, $T_k(x) < T_{k+1}(x)$ and $\phi_{T_k(x)}(x) = f^k(x)$. Then the following statements are equivalent:*

1. *The flow ϕ is kinematic expansive.*
2. *There is $\delta > 0$ such that if $\operatorname{dist}(\phi_t(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ with $x, y \in l$ then $x = y$.*
3. *There is $\rho > 0$ such that if $x, y \in l$, $\operatorname{dist}(f^n(x), f^n(y)) < \rho$ and $|T_n(x) - T_n(y)| < \rho$ for all $n \in \mathbb{Z}$ then $x = y$.*

Proof. $(1 \to 2)$. Let $\varepsilon > 0$ be such that if $x \in l$ and $0 < |s| < \varepsilon$ then $\phi_s(x) \notin l$. Since ϕ is kinematic expansive there is an expansive constant $\delta > 0$ associated to ε. Take $x, y \in l$ such that $\operatorname{dist}(\phi_t(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$. Then there exists $s \in (-\varepsilon, \varepsilon)$ such that $y = \phi_s(x)$. But this implies that $s = 0$ and $x = y$.

$(2 \to 3)$. Let $T^* = \max\{T_1(x) : x \in l\}$. The continuity of the flow implies that there exists $\delta' > 0$ such that:

$$\text{if } \operatorname{dist}(x, y) < \delta' \text{ then } \operatorname{dist}(\phi_t(x), \phi_t(y)) < \delta \text{ for all } t \in [0, T^*]. \tag{6.1}$$

By the triangular inequality we have that:

$$\operatorname{dist}(\phi_{T_k(x)}(x), \phi_{T_k(x)}(y)) \le \operatorname{dist}(f^k(x), f^k(y)) + \operatorname{dist}(\phi_{T_k(x)}(y), \phi_{T_k(y)}(y)) \tag{6.2}$$

for all $x, y \in l$ and $k \in \mathbb{Z}$. We will show that $\rho = \delta'/2$ satisfies the thesis. Assume that $x, y \in l$, $\operatorname{dist}(f^n(x), f^n(y)) < \rho$ and $|T_n(x) - T_n(y)| < \rho$ for all $n \in \mathbb{Z}$. By inequality (6.2) we have that $\operatorname{dist}(\phi_{T_k(x)}(x), \phi_{T_k(x)}(y)) \le \delta'$ for all $n \in \mathbb{Z}$. Now, applying condition (6.1) we have that $\operatorname{dist}(\phi_t(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ and therefore, $x = y$ because $x, y \in l$.

$(3 \to 1)$ Let $\rho > 0$ be the constant given in item (3). Given $\varepsilon > 0$ consider $\delta > 0$ such that if $\operatorname{dist}(x, y) < \delta$ with $x \in l$ and $y \in X$ then

$$\text{there is a unique } s \in \mathbb{R} \text{ such that } |s| < \varepsilon, |s| < \rho \text{ and } \phi_s(y) \in l \cap B_\rho(x). \tag{6.3}$$

This value of s will be denoted as $s_x(y)$ and we define the projection $\pi_x\colon B_\rho(x) \to l$ as $\pi_x(y) = \phi_{s_x(y)}(y)$. We will show that δ is an expansive constant associated to ε. Suppose that $\operatorname{dist}(\phi_t(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$. Without loss of generality we assume that $x \in l$. Define the sequence $y_n = \phi_{T_n(x)}(y)$ for $n \in \mathbb{Z}$. We have that $f^n(y_0) = \pi_{f^n(x)}(y_n)$ and also $\operatorname{dist}(f^n(x), y_n) < \delta$ for all $n \in \mathbb{Z}$. By condition (6.3) for each $n \in \mathbb{Z}$ there is s_n such that $|s_n| < \rho$, $\phi_{s_n}(y_n) = f^n(y_0)$ and $\operatorname{dist}(f^n(x), f^n(y_0)) < \rho$ for all $n \in \mathbb{Z}$. If we apply our hypothesis to the points $x, y_0 \in l$, noting that $|s_n| = |T_n(x) - T_n(y_0)|$, we conclude that $x = y_0$. Therefore $x = \phi_{s_0}(y)$, and since $|s_0| < \varepsilon$ by (6.3), the proof ends. □

As an application of this result we have that the flow on Example 6.1.4 (periodic band) is kinematic expansive. Note that this is a suspension of the identity map of an arc under an increasing return time function. In the next section we will prove that the interval is the only connected space whose identity map admits a kinematic expansive suspension.

6.3.2 Suspensions of the identity map

In general topology it is an important task to give intrinsic topological characterizations of topological spaces. For example, it is known that a compact metric space X is homeomorphic to the usual Cantor set if and only if it is totally disconnected (every component is trivial) and perfect (no isolated points). From a dynamical viewpoint it is also possible to characterize topological spaces. Let us mention, as an example, that a compact surface is a torus if and only if it admits an Anosov diffeomorphism. Finite sets can be characterized as those admitting a positive expansive homeomorphism.

In this section we give a dynamical characterization of compact metric spaces that can be embedded in $\mathbb{R}$. In order to obtain this kind of result we recall a topological characterization of such spaces.

Theorem 6.3.2. *A compact metric space l is homeomorphic to a subset of $\mathbb{R}$ if and only if the following statements hold:*

1. *the components of l are points or compact arcs,*
2. *no interior point of an arc-component a is a limit point of $l \setminus a$ and*
3. *each point of l has arbitrarily small neighborhoods whose boundaries are finite sets.*

See [113] for a proof.

Theorem 6.3.3. *If l is a compact metric space then the following statements are equivalent:*

1. *the identity map of l admits a kinematic expansive suspension,*
2. *there is a continuous and locally injective map $T\colon l \to \mathbb{R}$, i.e., there is $\delta > 0$ such that if $0 < \operatorname{dist}(x, y) < \delta$ then $T(x) \neq T(y)$ and*

3. l is homeomorphic to a subset of $\mathbb{R}$.

Proof. ($1 \to 2$) The return time map T making the suspension of the identity map of l kinematic expansive, has to be locally injective by Proposition 6.3.1 (item 3).

($2 \to 3$) Since l is compact we have that T is a local homeomorphism. Therefore l satisfies item (3) of Theorem 6.3.2. To prove the first item, consider a non trivial component a of l. As we mentioned, T is a local homeomorphism, therefore a is a compact connected one-dimensional manifold. If a is not a compact arc, then it must be a circle, but this easily gives us that T cannot be locally injective. Therefore item (1) holds. The second item of Theorem 6.3.2 follows again because T is a local homeomorphism.

($3 \to 1$) Let $T\colon l \to \mathbb{R}^+$ be an embedding of l. Applying Proposition 6.3.1 we have that the suspension of the identity of l under T is kinematic expansive. □

Let Γ be the set of periodic orbits of ϕ endowed with the relative topology induced by the Hausdorff distance between compact subsets of X. Recall that

$$\text{dist}_H(A,B) = \inf\{\varepsilon > 0 : B \subset B_\varepsilon(A), A \subset B_\varepsilon(B)\}$$

is the Hausdorff distance between the compact sets $A, B \subset X$. Let $T\colon \Gamma \to \mathbb{R}^+$ be the period function defined such that $T(\gamma)$ is the period of the periodic orbit γ. The following proposition gives another characterization of the suspensions of the previous theorem.

Lemma 6.3.4. *If ϕ is kinematic expansive on a compact metric space then the period function T is continuous.*

Proof. Let γ_n be a sequence of periodic orbits converging in the Hausdorff distance to a periodic orbit γ. Let l be a local cross section through a point $p \in \gamma$. If $T(\gamma_n)$ do not converge to $T(\gamma)$ then γ_n, for large n, must meet at least twice to l, say in x_n and y_n. Therefore, x_n and y_n contradict the kinematic expansiveness of ϕ. □

Proposition 6.3.5. *Suppose that ϕ is a kinematic expansive flow without singularities on a compact metric space such that every orbit is compact. Then it is a suspension of the identity map of a compact subset of $\mathbb{R}$.*

Proof. By Lemma 6.3.4 and Theorem 6.3.3 we have that every point has a local cross section homeomorphic to a compact subset of $\mathbb{R}$. Therefore, the connected component of every point $x \in X$ is homeomorphic to an annulus A (possibly reduced to the orbit of x alone). Taking local cross sections through points in the boundaries of A, we can extend a global section of A. Therefore, every point admits a compact local cross section l (not necessarily small) such that $\phi_{\mathbb{R}}(l)$ is an open subset of X. With the techniques of [17] it is easy to prove that ϕ is a suspension. Therefore we conclude by Theorem 6.3.3. □

6.3.3 Arc homeomorphisms

In this section we study when a homeomorphism of a compact arc I admits a kinematic expansive suspension. We consider homeomorphisms of class C^0 and C^1.

Given a continuous flow $\phi\colon \mathbb{R}\times X\to X$ we say that $A\subset X$ is *positively invariant* if for all $x\in A$ and for all $t\geq 0$ it holds that $\phi_t(x)\in A$. Recall that the *ω-limit set* of x is

$$\omega(x)=\{y:\exists t_k\to+\infty \text{ such that } \phi_{t_k}(x)\to y \text{ as } k\to+\infty\}.$$

Lemma 6.3.6. *Let ϕ be a continuous flow with a positively invariant annulus A such that one component of the boundary is a periodic orbit γ, the other component is transversal to the flow and the ω-limit set of every point in A is γ. Then ϕ restricted to A admits a kinematic expansive time change.*

Proof. Consider a global cross section l, as in Figure 6.8, and identify l with the interval $[0,1]$. The return map to l is conjugated with $f\colon[0,1]\to[0,1]$ defined by $f(x)=x/2$. Then $f^n(x)=x/2^n$ for all $n\geq 0$ and $x\in[0,1]$. Define $a_n=f^n(1)$ and $b_n=f^n(1/2+1/2^{n+2})$. In this way we have that $b_n\in(a_n,a_{n+1})$ for all $n\geq 0$. Define $T\colon[0,1]\to\mathbb{R}$ by $T(a_n)=T(1)=T(0)=1$ and $T(b_n)=1+1/(n+1)$ for all $n\geq 0$ and extended by linearity in (a_n,b_n) and (b_n,a_{n+1}) for all $n\geq 0$. See Figure 6.9.

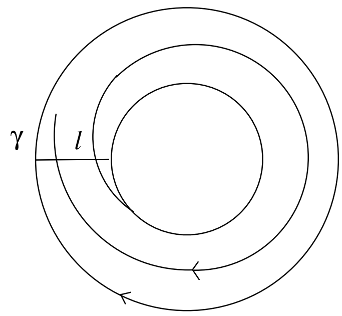

Figure 6.8: .

Consider a semi-flow ψ, a time change of ϕ with returning time T to the section l. We will show that ψ is kinematic expansive. Every point in γ is separated from any other outside γ, as can be easily seen. We now study two cases, taking $x,y\in l=[0,1]$, $x\neq y$.

Case 1: $a_1<x<y\leq a_0$. Notice that there exists n_0 such that if $n\geq n_0$ then $a_{n+1}<b_n<x_n<y_n<a_n$, being $x_n=f^n(x)$ and $y_n=f^n(y)$. Then for all $n\geq n_0$ we have that:

$$T(x_n)-T(y_n)\geq\frac{(y_n-x_n)n^{-1}}{a_{n+1}-a_n}=\frac{\left(\frac{y-x}{2^n}\right)n^{-1}}{1/2^{n+1}}=2(y-x)/n.$$

And then $\sum_{i=0}^{\infty}T(x_i)-T(y_i)=+\infty$ and therefore the points x,y are separated by the flow ψ.

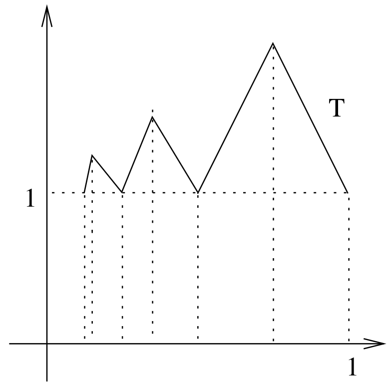

Figure 6.9: The return time function T.

Case 2: $a_0 < x \leq a_1 < y < a_2$. From the definition of T it is easy to see that

$$T(x_n) = \frac{2(a_1 - x)}{n(1 - 1/2^{n+1})}$$

and

$$T(y_n) = \frac{2(a_2 - y)}{(n+1)(1 - 1/2^{n+2})}.$$

Let $\alpha_n = \frac{2(a_1-x)}{1-1/2^{n+1}}$ and $\beta_n = \frac{2(a_2-y)}{1-1/2^{n+2}}$. Then

$$T(x_n) - T(y_n) = \frac{\alpha_n}{n} - \frac{\beta_n}{n+1} = \frac{\alpha_n - \beta_n}{n+1} - \frac{\alpha_n}{n(n+1)}$$

Again we have that $\sum_{n=0}^{\infty} T(x_n) - T(y_n) = +\infty$ since $\alpha_n - \beta_n \to 1/2 + 2(y - x) \neq 0$. □

Recall that for an arc homeomorphism preserving orientation, the periodic points are in fact fixed points, and given any closed set F of the arc there is an orientation preserving homeomorphism whose set of fixed points is F. If f reverses orientation we have that there is a unique fixed point and other periodic points have period 2.

Proposition 6.3.7. *A homeomorphism $f\colon I \to I$ admits a kinematic expansive suspension if and only if the set of periodic points has finitely many components and the period function is continuous (i.e. in the reversing orientation case, the fixed point is not accumulated by points of period 2).*

Proof. ($\Rightarrow$) Let us start assuming that f admits a kinematic expansive suspension. Suppose first that f reverses orientation. As we said f has a unique fixed point p. Now, it is easy to see that x and $f(x)$ contradicts expansiveness if x is a periodic point (of period 2) arbitrarily close to p. Assume now that there are infinitely many wandering components. We have that for all $\varepsilon > 0$ there is a wandering point x such that $\operatorname{dist}(x, f^n(x)) < \varepsilon$ for all $n \in \mathbb{Z}$. Consider a time map $T\colon I \to \mathbb{R}^+$. Since it is uniformly continuous we have that for all $\delta > 0$, the value of ε can be chosen in such a way that if $\mathrm{dist}(x, y) < \varepsilon$ then $|T(x) - T(y)| < \delta$. Therefore, the points x and $f(x)$ contradicts the expansiveness.

($\Leftarrow$) On each component of fixed points consider an increasing time map. On wandering points use Lemma 6.3.6. □

The smooth case is very restrictive as the following result shows.

Proposition 6.3.8. *Assume that $f: I \to I$ is a homeomorphism and $T: I \to \mathbb{R}^+$ is C^1. If the suspension (f, T) is kinematic expansive then f is the identity and T is strictly increasing or decreasing.*

Proof. Let us assume first that f is increasing. By contradiction assume that it is not the identity, therefore there are two fixed points $p, q \in I$ such that for all $x \in (p, q)$ we have that $f^n(x) \to q$ and $f^{-n}(x) \to p$ as $n \to \infty$. Since T is smooth we have that $T(y) - T(x) = \int_x^y T'(u)\, du$. Therefore, taking $x, y \in (p, q)$ arbitrarily close we can easily contradict Proposition 6.3.1.

Assume now that f is decreasing and take the fixed point p of f. If close to p there are wandering points then we can arrive to a contradiction as in the previous case. The other possible case is that every point close to p is periodic with period 2. If x is close to p and $y = f(x)$ it is easy to see that x, y contradicts the expansiveness of the suspension flow. This contradiction proves that $f(x) = x$ for all $x \in I$.

Now applying Proposition 6.3.1 we see that T must be increasing or decreasing. □

6.3.4 Circle homeomorphisms

Let $f: S^1 \to S^1$ be homeomorphism of the circle. Recall that if the are no wandering points then it is conjugated to a rotation. In other case we say that the wandering set of f is *finitely generated* if there is a finite number of disjoint open arcs $a_1, \dots, a_n$ such that the wandering set is the union

$$\bigcup_{j \in \mathbb{Z}, i=1,\dots,n} f^j(a_i).$$

In the following Theorem we exclude the case where f is minimal because we have no C^0 general answer.

Theorem 6.3.9. *A non-minimal circle homeomorphism $f: S^1 \to S^1$ preserving orientation admits a kinematic expansive suspension if and only if its wandering set is non-empty and finitely generated.*

Proof. ($\Rightarrow$) Assume that f admits a kinematic expansive suspension. If f has no wandering points then it is a rotation, and since it is not minimal, it is a periodic (rational) rotation.

Now it is easy to see that there are arbitrarily close points with the same period (for the flow) contradicting expansiveness. Therefore the wandering set is not empty. The wandering set is finitely generated by the arguments in the proof of Proposition 6.3.7.

($\Leftarrow$) If the the rotation number of f is rational the proof is reduced to Proposition 6.3.7. Therefore, we will assume that f has irrational rotation number. Also assume that the wandering set is generated by one interval (it is easy to extend the proof to the general case). It is known that $f\colon \Omega \to \Omega$ is an expansive homeomorphism, where Ω denotes the non-wandering set of f. Assume that the wandering set is the disjoint union $\cup_{n\in\mathbb{Z}} f^n(I)$ where $I = (a, b)$ is an open arc. Without loss of generality we will assume that

$$\operatorname{dist}(f^n(x), f^n(y)) = \frac{\operatorname{dist}(x, y)}{2^n} \tag{6.4}$$

for all $x, y \in I$ and $n \geq 0$. For each $n \geq 0$ take a point $z_n \in f^n(I)$ such that $f^{-n}(z_n) \to a$.

Define a continuous map $T\colon S^1 \to \mathbb{R}^+$, the return time function, as $T(x) = 1$ if $x \in \Omega$ or $x \in \cup_{n\geq 0} f^{-n}(I)$, $T(z_n) = 1 + 1/n$ for all $n > 0$ and extend T linearly on each $f^n(I)$ with $n > 0$.

We claim that the flow on the torus with return map f and return time T, defined above, is kinematic expansive. To prove kinematic expansiveness we will use item (3) of Proposition 6.3.1. We know that $f\colon \Omega \to \Omega$ is an expansive homeomorphism. It is easy to see that if $x \in I$ and $y \notin \operatorname{clos}(I)$ then x, y are separated by f. It only rests to consider $x, y \in \operatorname{clos}(I)$. We divide the proof in two cases.

First suppose that $x, y \in I$. In the arc I we consider an order such that $a < b$ and using the homeomorphism f we induce an order on each $f^n(I)$ with $n \in \mathbb{Z}$. Assume that $x < y$. Recall that the sequence $z_n \in f^n(I)$ used to define the return time T has the property $f^{-n}(z_n) \to a$. Therefore there is $n_0 \geq 0$ such that $z_n < f^n(x) < f^n(y)$ for all $n \geq n_0$. Let us introduce the notation $x_n = f^{n+n_0}(x)$ and $y_n = f^{n+n_0}(y)$. By the definition of T (recall that it was extended linearly) and equation (6.4) we have that

$$\begin{aligned} T(x_n) - T(y_n) &\geq \frac{\operatorname{dist}(x_n, y_n)}{(n_0 + n)\operatorname{dist}(f^{n+n_0}(a), f^{n+n_0}(b))} \\ &= \frac{\operatorname{dist}(x_{n_0}, y_{n_0})}{(n_0 + n)\operatorname{dist}(f^{n_0}(a), f^{n_0}(b))} \end{aligned}$$

for all $n \geq 0$. Then

$$\sum_{n\geq 0} T(f^n(x)) - T(f^n(y)) = \infty.$$

Now assume that $x = a$ (a extreme point of I) and $y \in I$. Assume that $z_n < f^n(y)$ for all $n \geq n_0$. As before it can be proved that

$$T(y_n) - T(x_n) \geq \frac{1}{n}\frac{\operatorname{dist}(f^{n_0}(b), f^{n_0}(y))}{\operatorname{dist}(a, b)}.$$

And we arrive again to a divergent series. The case $y = b$ is similar to this case. This proves that the flow is kinematic expansive. □

Theorem 6.3.10. *An orientation reversing homeomorphism $f\colon S^1 \to S^1$ admits a kinematic expansive suspension if and only if it has wandering points, fixed points are not accumulated by periodic points and the wandering set has a finite number of components.*

Proof. Since f reverses orientation it has two fixed points. The dynamics is then reduced to an interval homeomorphism and we can apply Proposition 6.3.7 to conclude the proof. □

6.3.5 Smooth suspensions of circle diffeomorphisms

In this section we apply the results of [11] to study smooth kinematic expansive suspensions of irrational rotations.

Theorem 6.3.11. *No suspension of an irrational rotation $f\colon S^1 \to S^1$ with C^1 return time function $T\colon S^1 \to \mathbb{R}^+$ is kinematic expansive.*

Proof. Let μ denote the f-invariant Lebesgue probability measure on the circle. Define

$$\tau = \int_{S^1} T\, d\mu.$$

Denote by $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ the angle of the rotation f. Let $q_n \in \mathbb{N}$ be the denominator of of the n^{th}-convergent of the rational approximation of α by the continued fraction algorithm. It holds that $f^{q_n}(x) \to x$ as $n \to \infty$ for all $x \in S^1$. See, for example, Section 2.3.2 of [11] for more details. Consider the Birkhoff sum

$$T_m(x) = \sum_{i=0}^{m-1} T(f^i(x)).$$

The improved Denjoy-Koksma Theorem proved in [11] states that

$$\sup_{x \in S^1} |\tau q_n - T_{q_n}(x)| \to 0 \tag{6.5}$$

as $n \to \infty$. Fix $x_0 \in S^1$ and define $x_n = f^n(x_0)$ for all $n \geq 0$. It is easy to see that $T_{m+n}(x) = T_m(x_n) + T_n(x_0)$. Then

$$T_n(x_m) - T_n(x_0) = T_m(x_n) - T_m(x_0)$$

and in particular

$$T_{q_n}(x_m) - T_{q_n}(x_0) = T_m(x_{q_n}) - T_m(x_0)$$

for all $m, n \geq 0$. Applying equation (6.5) we have that for all $\varepsilon > 0$ there is N such that

$$|T_{q_n}(x_m) - T_{q_n}(x_0)| < \varepsilon$$

for all $n \geq N$ and $m \geq 0$. Therefore

$$|T_m(x_{q_n}) - T_m(x_0)| < \varepsilon$$

for all $n \geq N$ and $m \geq 0$. Now we can take $n \geq N$ such that the distance between x_{q_n} and x_0 is smaller than ε. Therefore these two points are not separated by the suspension flow in positive time.

Finally, considering f^{-1} instead of f and arguing as above we conclude that these two points are not separated by the flow in negative times. Therefore, the flow is not kinematic expansive. □

Question 6.3.12. *Are there C^0 minimal kinematic expansive flows on the torus?*

6.4 Kinematic expansive flows on surfaces

In Section 6.2 we studied flows with the property of having every time change being kinematic expansive (strong kinematic expansiveness). In this section we consider what could be called *conditional expansiveness*: the kinematic expansiveness of the flow depends on the time change. We consider flows on the disc and the annulus. In the final subsection we prove that every compact surface admits a kinematic expansive flow.

6.4.1 The disc

Let D be a two-dimensional compact disc and consider $\phi\colon \mathbb{R} \times D \to D$ a continuous flow. It is well known that under these conditions, ϕ has a singular point. For a kinematic expansive flow we show that at least one singularity must be in the interior of the disc. Next we study the relationship between the number of singularities and the differentiability of the flow.

Proposition 6.4.1. *If ϕ is a kinematic expansive flow on a disc D then ϕ has a singularity in the interior of D.*

Proof. Assume by contradiction that the singularities are in the boundary ∂D. The α and ω-limit set of every point of D must be a singular point, it follows by Poincaré-Bendixon Theorem. Consequently, there must be two singular points p and q such that the set U of points $x \in D$ such that $\alpha(x) = \{p\}$ and $\omega(x) = \{q\}$ has non-empty interior. Now it is easy to see that two points close to an interior point of U contradict the expansiveness of the flow. □

The following result proves that the disc admits kinematic expansive flows. In particular this flow may have just one singular point.

Proposition 6.4.2. *Suppose that ϕ is a continuous flow in D with a finite number of singularities and $p \in D$ is an interior point. Assume that p is a repeller fixed point and for all $x \neq p$ interior to D the ω-limit set of x is $\omega(x) = \partial D$. Then ϕ admits a kinematic expansive time change.*

Proof. Let l be a local cross section of the flow meeting the boundary of D. Suppose that the return map on l is the continuous map $f : l \to l$ and the return time is $T : l \to \mathbb{R}^+$. If there are no singular point in the boundary then we can apply Lemma 6.3.6 to conclude. Therefore we will assume that there are singularities in the boundary and l is as in Figure 6.10, where q is another singular point.

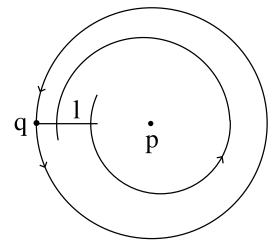

Figure 6.10: Kinematic expansive flow on the disc.

Without loss of generality we assume that the return map f is $f(x, 0) = (x/2, 0)$. Consider a time change such that the return map of l is $T(x, 0) = 1/x$. Given $1/2 \leq x < y < 1$ we define $x_n = f^n(x) = x/2^n$ and $y_n = f^n(y) = y/2^n$. Then

$$T(y_n) - T(x_n) = 2^n \left(\frac{1}{x} - \frac{1}{y} \right)$$

and therefore $\sum_{i=0}^{\infty} T(y_i) - T(x_i) = +\infty$. It implies that ϕ is kinematic expansive. □

The previous result does not hold if we add a hypothesis of differentiability.

Theorem 6.4.3. *If ϕ is a smooth kinematic expansive flows in the disc then ϕ has at least two singular points.*

Proof. By contradiction assume that ϕ has only one singularity $p \in D$. By Proposition 6.4.1 we know that the singular point is in the interior of D and therefore ∂D is a periodic orbit. Since there is just one singular point, the periodic orbits in D can be totally ordered with respect to the interior singular point (i.e., if γ_1, γ_2 are periodic orbits then $\gamma_1 < \gamma_2$ if γ_1 separates p from γ_2). Considering a minimal periodic orbit, we obtain a sub-disc $D' \subset D$, bounded by such minimal periodic orbit, such that in the interior of D' there is no periodic orbit. Now, applying the techniques of Proposition 6.3.8, near $\partial D'$, we arrive to a contradiction. □

6.4.2 Periodic bands

Denote by $A \subset \mathbb{R}^2$ a compact annulus bounded by two circles centered at the origin.

Proposition 6.4.4. *Suppose that ϕ is a kinematic expansive flow on A such that every orbit is contained in a circle centered at the origin. If ϕ has singular points then they all are in one of the components of the boundary. In particular there are no interior singular points.*

Proof. We know that the set of singular points is finite. Let us first show that there is no singularity in the interior. We argue by contradiction. Take a segment l transversal to the flow meeting at p the circle of an interior singularity. We can assume that there are no singular points in the circles of any $q \in l$ if $q \neq p$. Since the circle of p has at least one singularity we have that the return time map of $l \setminus \{p\}$ diverges to $+\infty$ at p. Therefore we can find two points, as close to p as we wish, in different components of $l \setminus \{p\}$ with the same period. These points contradict kinematic expansiveness.

Now assume that there are singular points in both components of ∂A. Let s be a global cross section of the flow meeting once each interior orbit. As before, the return time map T diverges in the boundaries of s. Since T is continuous, it has a minimum at some interior point $x \in s$. Now we can find two points in different components of $l \setminus \{x\}$ with the same period. If these points are sufficiently close to x, then kinematic expansiveness can be contradicted for arbitrary small expansive constants. □

Remark 6.4.5. *Notice that we have considered kinematic expansive flows on the annulus in Section 6.3.3 (i.e., suspensions of increasing arc homeomorphisms).*

6.4.3 Every compact surface admits a kinematic expansive flow

As mentioned in Remark 6.2.9, there are surfaces do not admitting strong kinematic expansive flows. For kinematic expansiveness there is no such restriction.

Theorem 6.4.6. *Every compact surface admits a kinematic expansive flow.*

Proof. Given a compact surface S consider a triangulation $T_1, \dots, T_n$. Fix an orientation on each edge. In Figure 6.11 we see that each triangle T_i admits a kinematic expansive flow (recall Proposition 6.4.2) with any prescribed orientation in the edges and singular points in the corners. Now it is easy to see that the global flow is kinematic expansive. □

6.5 Hyper-expansive flows

Let $\phi\colon \mathbb{R} \times X \to X$ be a continuous flow on a compact metric space X. Denote by $\mathbb{K}$ the space of compact subsets of X with the Hausdorff metric dist_H. Denote by $\phi^*\colon \mathbb{R} \times \mathbb{K} \to \mathbb{K}$ the flow induced by ϕ, that is: $\phi^*_t(A) = \phi_t(A)$ for every compact set $A \in \mathbb{K}$.

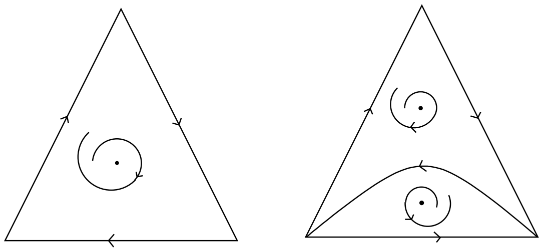

Figure 6.11: A kinematic expansive flow on each triangle.

Proposition 6.5.1. *If ϕ^* is expansive (in the sense of Bowen-Walters) then X is a finite set.*

Proof. Assume by contradiction that $|X| = \infty$. It is easy to see that this implies that there is at least one regular orbit of ϕ. So, consider such a regular point $x \in X$. Given $\delta > 0$ consider $\varepsilon > 0$ such that $\operatorname{diam}(\phi_{[0,\varepsilon]}(x)) < \delta$ for all $t \in \mathbb{R}$. Define $A = \{x, \phi_\varepsilon(x)\}$ and $B = \{x, \phi_{\varepsilon/2}(x)\}$. We have that $\operatorname{dist}_H(\phi_t^*(A), \phi_t^*(B)) < \delta$ for all $t \in \mathbb{R}$. It also holds that $\phi_t^*(A) \neq B$ for all $t \in \mathbb{R}$. Therefore ϕ^* is not expansive and the proof ends. $\square$

Chapter 7

Positive expansive flows

In this chapter we consider positive expansive flows from a kinematic and a geometric viewpoint.

In Section 7.1 the case of positive kinematic expansiveness is considered. Basic examples are shown and we study the local behavior of the flow near a compact orbit. On surfaces, we prove that positive expansive flows are suspensions and has no singularities. The smooth case is also considered. We consider a variation of an example in [67] to show (on a compact metric space) that a positive kinematic expansive flow may not be negative kinematic expansive.

In Section 7.2 we prove that positive geometric expansive flows consists in a finite number of compact orbits (singular or periodic).

7.1 Positive kinematic expansive flows

In this section we consider positive expansive flows.

Definition 7.1.1. A flow ϕ is *positive kinematic expansive* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\operatorname{dist}(\phi_t(x), \phi_t(y)) < \delta$ for all $t \geq 0$ then there exists $s \in \mathbb{R}$ such that $y = \phi_s(x)$ and $|s| < \varepsilon$.

Remark 7.1.2. *If X is a compact subset of $\mathbb{R}$ then for every injective and continuous map $T \colon X \to \mathbb{R}^+$ the suspension flow of the identity map $f \colon X \to X$ by T is positive kinematic expansive. Notice also that it is negative expansive, i.e., its inverse flow is positive expansive.*

In this section we first study the behavior of a positive kinematic expansive (and also separating) flow near a compact orbit. On surfaces we give a characterization of such flows. We also show that on a compact metric space a positive kinematic expansive flow may not be negative kinematic expansive.

7.1.1 Periodic orbits

In this section we consider compact orbits of positive kinematic expansive and separating flows.

Proposition 7.1.3. *Let ϕ be a positive kinematic expansive flow on a compact metric space. If γ is a periodic orbit then there exists $\delta > 0$ such that if $\phi_{\mathbb{R}^+}(x) \subset B_\delta(\gamma)$ and $x \notin \gamma$ then x is a periodic point.*

Proof. Let $l \subset X$ be a small local cross section of time $\tau > 0$ meeting the periodic orbit γ only at some point $p \in \gamma$. Assume that there is $x \in l$ such that $\phi_t(x)$ is close to γ for all $t \geq 0$. Let x_n be the sequence of returns of x to l and consider the increasing sequence of return times t_n such that $\phi_{t_n}(x) = x_n$ with $x_0 = x$. Take $y = x_1$. Denote by T the return time map of l. Consider $k = \sup_{a,b \in l} |T(a) - T(b)|$. Notice that $k \to 0$ if $\operatorname{diam}(l) \to 0$. Denote by f the first return map of l, $f(a) = \phi_{T(a)}(a)$ for all $a \in l$ where f is defined. Since $y = f(x)$ we have that

$$\left| \sum_{i=0}^{n} T(f^i(y)) - \sum_{i=0}^{n} T(f^i(x)) \right| = |T(f^{n+1}(x)) - T(x)| \leq k$$

for all $n \geq 0$.

Therefore, expansiveness implies that x and y are in the same local orbit and since x, y are in the local cross section l we have that $y = x$. Then x is a fixed point of f and a periodic point of ϕ. □

Definition 7.1.4. A flow ϕ is *positive separating* if there exists $\delta > 0$ such that if

$$\operatorname{dist}(\phi_t(x), \phi_t(y)) < \delta$$

for all $t \geq 0$ then $y = \phi_{\mathbb{R}}(x)$.

The following example is a positive separating flow that is not positive kinematic expansive and it shows that Proposition 7.1.3 does not hold for positive separating flows.

Example 7.1.5. *Let $X = \{0, 1\} \cup \{x_n : n \in \mathbb{Z}\} \subset \mathbb{R}$ such that x_n is an increasing sequence, $\lim_{n \to -\infty} x_n = 0$ and $\lim_{n \to \infty} x_n = 1$. Define the homeomorphism $f \colon X \to X$ by $f(0) = 0$, $f(1) = 1$ and $f(x_n) = x_{n+1}$. Consider $T \colon X \to \mathbb{R}^+$ given by $T(0) = T(1) = 1$ and $T(x_n) = \frac{1}{|n|+1}$ for all $n \in \mathbb{Z}$. Let ϕ be the suspension flow of f by T. By the previous proposition it is easy to see that it is not positive kinematic expansive. It also holds that ϕ is positive separating (the proof is trivial because there are only three orbits for the flow) and it shows that Proposition 7.1.3 does not hold for positive separating flows.*

Proposition 7.1.6. *Suppose that ϕ is a positive separating flow with a singular point $p \in X$. If for some $x \in X$ it holds that $p \in \omega(x)$ then $x = p$. Consequently, there are no singularities in the ω-limit set of a regular point.*

Proof. Arguing by contradiction consider $x \neq p$ with $p \in \omega(x)$. Since the flow is positive separating, there is $\delta > 0$ such that if $y \in B_\delta(p)$ and $y \neq p$ then there is $t > 0$ such that

$\phi_t(y) \notin B_\delta(p)$. Therefore, there are two increasing and divergent sequences $t_n, s_n \in \mathbb{R}$ such that $t_n < s_n$ for all $n \geq 1$, $s_n - t_n \to \infty$, $\phi_{s_n}(x) \to p$, $\phi_{t_n}(x) \in \partial B_\delta(p)$ and $\phi_{[t_n,s_n]}(x) \subset B_\delta(p)$. If z is a limit point of $\phi_{t_n}(x)$ it is easy to see that $\phi_{\mathbb{R}^+}(z) \subset B_\delta(p)$. But this contradicts that ϕ is positive separating. □

7.1.2 Positive kinematic expansive flows on surfaces

In this section we classify positive kinematic expansive flows of compact surfaces. We consider the C^0 and C^2 case.

Lemma 7.1.7. *Let $l = [a, b]$ and l' be two compact local cross sections and suppose there exists a continuous non-bounded function $\tau\colon [a, b) \to \mathbb{R}$ such that $\phi_{(0,\tau(x))}(x) \cap l = \emptyset$ and $\phi_{\tau(x)}x \in l'$ for all x in $[a, b)$. Then $\omega(b) \subset Sing$.*

Proof. See Lemma 3 in [103] or Lemma 2.2 in [5]. □

Proposition 7.1.8. *If ϕ is positive kinematic expansive on a compact surface then $Sing(\phi) = \emptyset$.*

Proof. By contradiction assume that $p \in S$ is a singular point. By Proposition 7.1.6 we have that p must be a repeller. Consider the open set

$$U = \left\{ x \in S : \lim_{t \to -\infty} \phi_t(x) = p \right\}.$$

We will show that ∂U is a periodic orbit. Take $x \in U$, $x \neq p$. Consider $y \in \omega(x)$. By Proposition 7.1.6 we have that y is a regular point. Let l be a compact local cross section with y as an extreme point. Assume that $\phi_{\mathbb{R}^+}(x)$ cuts l infinitely many times and denote by $x_1, x_2, \dots$ the cuts of the positive trajectory of x with l. Define

$$V = \{z \in l : \phi_{\mathbb{R}^+}(z) \cap l \neq \emptyset\}.$$

We will show that the arc $[x_1, x_2] \subset l$ is contained in V. Denote by V_1 the connected component of $V \cap [x_1, x_2]$ containing x_1. We have that V_1 is open in $[x_1, x_2]$. By Lemma 7.1.7 and Proposition 7.1.6 we have that the return time of the points in V_1 to l is bounded. Therefore, by the continuity of the flow and the compactness of l, the extreme points of V_1 are in V_1 and it is closed. This proves that $[x_1, x_2] \subset V_1$. Analogously it can be proved that $[x_n, x_{n+1}] \subset V$. Therefore every point in $l \setminus \{y\}$ returns to l. Again, if the return time were not bounded we contradict Proposition 7.1.6. Therefore y is a periodic point. But this is a contradiction with Proposition 7.1.3. Then, there cannot be singular points. □

Theorem 7.1.9. *Let ϕ be a continuous flow on a compact surface. If ϕ is positive kinematic expansive then it is topologically equivalent with one of the following models:*

1. *A suspension of the identity of* $[0, 1]$.
2. *A suspension of an orientation preserving circle homeomorphism with irrational rotation number and finitely generated wandering set.*

Proof. By Proposition 7.1.8 we only have to consider flows without singularities. It is known that the only surfaces admitting such flows are: the torus, the annulus, the Klein's bottle and the Moebius band. Suppose first that ϕ has a periodic orbit γ. If there is $x \notin \gamma$ such that $\phi_{-t}(x) \to \gamma$ as $t \to \infty$ then, arguing as in the proof of Proposition 7.1.8, we can prove that $\omega(x)$ is a periodic orbit. But this contradicts Proposition 7.1.3. Therefore, every orbit close to γ must be periodic. Now, applying Lemma 7.1.7 we have that every orbit is periodic because there are no singular points. Notice that γ must be two-sided, i.e., if U is a tubular neighborhood of γ then $U \setminus \gamma$ has two components. This is because, if this were not the case, then if T is the period of γ and x is close to γ then x and $y = \phi_T(x) \neq x$ would contradict kinematic expansiveness. Now recall that the Moebius band and the Klein bottle always have periodic orbits. Therefore S must be orientable. Also, the torus does not admit a kinematic expansive flow with every orbit being periodic. Therefore S must be an annulus.

Now suppose that S is the torus and ϕ has no periodic orbits. In this case it is known that ϕ is a suspension. Thus, we conclude by Theorem 6.3.9. □

Remark 7.1.10. *Concerning the converse of Theorem 7.1.9, we do not known if it is true if the flow is minimal. But assuming the conditions in (2) and that there are wandering points, we can define a positive kinematic expansive time change by applying the techniques of Theorem 6.3.9 above and Example 7.1.14 below.*

Theorem 7.1.11. *If* ϕ *is a* C^2 *positive kinematic expansive flow on a compact surface then* ϕ *is a suspension of the identity of* $[0, 1]$ *and* S *is an annulus.*

Proof. By Theorem 7.1.9 we have to show that ϕ cannot satisfy item (2) in this Theorem. By [41] and assuming that ϕ satisfies item (2) we have that ϕ is a minimal flow on the torus. By Theorem 6.3.11 we know that ϕ cannot be expansive. This contradiction proves the theorem. □

Let us introduce a natural definition.

Definition 7.1.12. A flow is *positive strong kinematic expansive* if every time change is positive kinematic expansive.

An example of such flow is the horocycle flow of a surface of negative curvature, this is proved in [40]. The horocycle flow is defined on three-dimensional manifold. We now apply our results to conclude that such flows do not exist on surfaces.

Corollary 7.1.13. *There are no positive strong kinematic expansive flows of surfaces.*

Proof. It follows by Proposition 7.1.8 and Theorem 6.2.8. □

7.1.3 Minimal positive expansiveness

In this section we consider an adaptation of an example in [67] to show that minimal positive kinematic expansive flows may not be trivial. We will suspend a minimal expansive homeomorphism of a Cantor set under a specific return time function. The example also shows that a positive kinematic expansive flow may not be negative expansive.

Example 7.1.14. *Let $\theta \in \mathbb{R}$ be an irrational number and consider the rotation $R\colon [0,1) \to [0,1)$ given by $R(x) = x + \theta \mod 1$. By* splitting *along the orbit of 0 under R we obtain a minimal expansive homeomorphism $f\colon l \to l$ on a Cantor set l. This homeomorphism is conjugate with a Sturmian subshift as defined in Section 3.7.1. Now let $x_n = R^n(0)$ and choose an increasing sequence of positive integers n_j such that x_{n_j} is strictly decreasing to 0. Next find a sequence δ_j decreasing to 0 such that, defining $I_j = [x_{n_j}, x_{n_j} + \delta_j]$, $I_j \cap I_k = \emptyset$ if $j \neq k$. Define a function $T\colon [0,1) \to \mathbb{R}^+$ by the conditions:*

1. *$T(x_{n_j}) = 1 + 1/j$ and $T(x_{n_j} + \delta_j) = 1$,*
2. *extend by linearity between the end points of each I_j and*
3. *$T(x) = 1$ otherwise.*

Note that since the discontinuities of T occurs at the points x_{n_j}, T can be extended to a continuous function on the Cantor set l.

Let ϕ be the suspension flow of $R\colon l \to l$ under the time function T. Given $x \in [0,1)$ in the orbit of 0 under R, denote by $x^+, x^- \in l$ the splitting points of x. Notice that

$$\operatorname{dist}(f^k(0^-), f^k(0^+)) \to 0$$

as $k \to \pm\infty$ and that there exists $\delta > 0$ such that if $x \notin \{f^i(0^\pm) : i \geq 0\}$ and $y \neq x$ then there is $k \geq 0$ such that $\operatorname{dist}(f^k(x), f^k(y)) > \delta$. If $T_n\colon l \to \mathbb{R}$ is given by $\phi_{T_n(x)}(x) = f^n(x)$ for all $x \in l$ then we have

$$T_{n_j}(0^-) - T_{n_j}(0^+) = \sum_{k=1}^{j} 1/k.$$

By Proposition 6.3.1 (the arguments in its proof) we conclude that ϕ is positive kinematic expansive. Note that $\operatorname{dist}(\phi_t(0^-), \phi_t(0^+)) \to 0$ as $t \to -\infty$ and then ϕ is not negative kinematic expansive.

7.1.4 Kinematic bi-expansive flows

In this brief section we wish to remark the non-existence of singularities for a flow being simultaneously positive and negative kinematic expansive. Let ϕ be a continuous flow on a compact metric space X and define the inverse flow ϕ^{-1} as $\phi^{-1}_t = \phi_{-t}$.

Definition 7.1.15. We say that ϕ is *kinematic bi-expansive* if ϕ and ϕ^{-1} are positive kinematic expansive.

Examples of such flows are the periodic annulus (Example 6.1.4) and the horocycle flow of a negatively curved surface [40].

Proposition 7.1.16. *If ϕ is a kinematic bi-expansive flow on a compact metric space X then every singularity is an isolated point of the space. Therefore, if X is connected with more than one point then there are no singularities.*

Proof. Assume by contradiction that p is a non-isolated singular point. Then, there is $x_n \to p$ with $x_n \neq p$ for all $n \geq 1$. Since the flow is positive kinematic expansive there are $\delta > 0$ and a divergent sequence $t_n > 0$ such that $\operatorname{dist}(\phi_{t_n}(x_n), p) = \delta$ and $\phi_{[0,t_n]}(x_n) \subset B_\delta(p)$. Taking z a limit point of $\phi_{t_n}(x_n)$ it is easy to prove that $\operatorname{dist}(\phi_{-t}(z), p) \leq \delta$ for all $t \geq 0$. Since δ can be taken arbitrarily small, we have that z and p contradict the negative kinematic expansiveness of the flow. This contradiction proves that singularities are isolated points of the space. □

7.2 Positive geometric expansive flows

As we proved in Theorem 3.2.3 (a well known result), if a compact metric space X admits a positive expansive homeomorphism then X is finite. In this section we show the corresponding result for flows, giving an affirmative answer to Problem 5.6 in [68]. We prove that if X admits a positive expansive flow (see Definition 7.2.8) then X is a finite union of circles and isolated points. No proof in the discrete case can be directly adapted to the case of flows because expansiveness of flows is defined using reparameterizations of time. So, new techniques are needed. In Section 7.2.1 we introduce a new metric, equivalent to the given one, that has regular properties in relation with the flow (see Proposition 7.2.7). This metric does not depend on the expansiveness of the flow and seems to be of interest on its own. It allows us to prove that every point of a positive expansive flow is negative Lyapunov stable (allowing a time reparameterization). Having proved that, we find again that the proofs in the discrete case cannot be adapted. For example, [75] considers a finite covering $U_1, \ldots, U_n$ such that $\operatorname{diam} f^{-k}(U_i) \to 0$ as $k \to \infty$ and easily it is concluded that the space is finite. In the continuous case one has to reparameterize the trajectories, so this argument is not easy to adapt for flows. The diameters of open sets never decrease to zero with a flow without singular points. And if one introduces reparameterizations, the flowed open sets may not cover. So, some care is needed to conclude the proof. A different argument is given in Section 7.2.4.

Let us remark some facts about *translating* results from discrete dynamics to flows. If $f\colon X \to X$ is a homeomorphism then stable sets are usually defined as

$$W^s_\gamma(x) = \{y \in X : \operatorname{dist}(f^n(x), f^n(y)) \leq \gamma \text{ for all } n \geq 0\}.$$

In the discrete-time case it is easy to prove that stable sets are closed sets. Consider $\phi\colon \mathbb{R}\times X \to X$ a continuous flow on a compact metric space. Since the definition of expansiveness for flows considers reparameterizations it is natural to define stable sets allowing reparameterizations: $y \in W^s_\gamma(x)$ if there is a reparameterization $h\colon \mathbb{R} \to \mathbb{R}$ (increasing homeomorphism with $h(0) = 0$) such that

$$\mathrm{dist}(\phi_{h(t)}(y), \phi_t(x)) \leq \gamma$$

for all $t \geq 0$. For flows, it is not true that $W^s_\gamma(x)$ is a closed set, see Example 9 in [126]. It seems to be a strange feature of the metric, so, it is natural to look for a new distance function (defining the same topology) with regular properties in relation with the flow (see Section 7.2.1).

In [78] it is proved that if X is a compact metric space admitting a minimal expansive homeomorphism then $\dim_{top}(X) = 0$. In the case of flows one would expect to conclude that $\dim_{top}(X) = 1$ if X admits an expansive minimal flow. In [67] techniques of local cross sections are used to attack the problem, but it remains an open case in order to give a complete translation (see Theorem 3.6 in [67]). This case is associated with *spiral orbits*. A point x is *spiral* if there is $t > 0$ such that $\phi_t(x) \in W^s_\gamma(x) \cap H(x)$ where $H(x)$ is a small local cross section of the flow at x. In the discrete case, spiral points give rise to periodic orbits, but in the case of flows it seems to be still an open problem.

In [121] (the last line of the first page) one finds the following: "Having first found theorems in the diffeomorphism case, it is usually a secondary task to translate the results back into the differential equations framework". In cited work, hyperbolic dynamics are considered. A special feature of hyperbolic flows is that stable sets need no reparameterizations. That is, if ϕ is a hyperbolic flow then:

1. for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\mathrm{dist}(\phi_{h(t)}(x), \phi_t(y)) < \delta$ for all $t \geq 0$ and some reparameterization h then $\mathrm{dist}(\phi_t(x), \phi_t(y)) < \varepsilon$ for all $t \geq 0$ and moreover,
2. for small s it holds that $\mathrm{dist}(\phi_{t+s}(x), \phi_t(y)) \to 0$ as $t \to \infty$.

It is a consequence of the Stable Manifold Theorem for hyperbolic flows. Therefore, techniques of diffeomorphisms can be adapted to flows, or at least they will not have to deal with reparameterizations. In light of this remark we wish to state the following problem. Given an expansive flow ϕ, find a topologically equivalent one ϕ' (i.e., both flows in X have the same orbits with the same orientation) satisfying items 1 and 2 above. The flow ϕ' can be considered as a global reparameterization of ϕ. If one is able to find such global reparameterization, stable sets would not need reparameterization and translations from expansive homeomorphisms to expansive flows would be easier. This seems to be an open problem. Another open problem is to define and construct a hyperbolic distance for an expansive flow as is done in [30] in the case of homeomorphisms.

Let us now describe the contents of this chapter. In Section 7.2.1 we define the Hausdorff metric for a flow and prove its main properties. In Section 7.2.2 some technical remarks are given related to expansiveness and reparameterizations. In Section 7.2.3 we show that the trajectories of a positive expansive flow are negative Lyapunov stable. In Section 7.2.4 we prove our main result and in Section 7.2.5 it is extended to positive expansiveness in the sense of Komuro [69] and singular points are allowed.

7.2.1 Hausdorff distance for a flow

In this section we consider a continuous flow on a compact metric space. We construct a metric that is equivalent with the original one and it has *good* properties relative to the flow.

Given a compact metric space (X, dist) consider the hyper-space $(\mathbb{K}, \operatorname{dist}_H)$. Let $\phi\colon X\times\mathbb{R} \to X$ be a continuous flow. Denote by I the real interval $[-1, 1]$ and for any $\tau > 0$ define $I\tau = [-\tau, \tau]$. Consider the map $\phi_{I\tau}\colon X \to \mathbb{K}$ that associates to each point its $I\tau$-orbit segment

$$\phi_{I\tau}(x) = \{\phi_t x : |t| \leq \tau\}.$$

Proposition 7.2.1. *For every $\tau > 0$ the map $\phi_{I\tau}$ is uniformly continuous.*

Proof. By the uniform continuity of the flow on compact intervals of time, we have that given $\varepsilon > 0$ there is $\delta > 0$ such that if $\operatorname{dist}(x, y) < \delta$ then $\operatorname{dist}(\phi_t x, \phi_t y) < \varepsilon$ for every $t \in I\tau$ and every $x, y \in X$. So $\operatorname{dist}_H(\phi_{I\tau}(x), \phi_{I\tau}(y)) < \varepsilon$ if $\operatorname{dist}(x, y) < \delta$. □

Notice that if the flow has periodic orbits with arbitrarily small periods then $\phi_{I\tau}$ cannot be injective. We do not consider singularities (i.e. equilibrium points) as periodic points.

Proposition 7.2.2. *The map $\phi_{I\tau}$ is injective if there are no periodic orbits of period smaller or equal than 3τ.*

Proof. Arguing by contradiction assume that $\phi_{I\tau}(x) = \phi_{I\tau}(y)$ with $x \neq y$. It implies that x is not singular. Without loss of generality we can assume that there is $s \in (0, \tau]$ such that $y = \phi_s(x)$. Then $\phi_{[s-\tau,s+\tau]}x = \phi_{[-\tau,\tau]}x$. So, $\phi_{s+\tau}x = \phi_{s'}x$ for some $s' \in I\tau$. Therefore $\phi_{s+\tau-s'}x = x$. This is a contradiction because $0 < s + \tau - s' \leq 3\tau$ and x is not singular. □

Notice that expansive flows (with or without singular points) and flows without singular points (expansive or not) do not have orbits with arbitrarily small periods.

Assuming that $\phi_{I\tau}$ is injective we consider the following distance in X

$$\operatorname{dist}(x, y) = \operatorname{dist}_H(\phi_{I\tau}(x), \phi_{I\tau}(y)).$$

Proposition 7.2.3. *If $\phi_{I\tau}$ is injective then the new distance* dist *is equivalent with* dist.

Proof. Since $\phi_{I\tau}$ is continuous and X is compact, the image of $\phi_{I\tau}$ is compact. So $\phi_{I\tau}\colon X \to \phi_{I\tau}(X)$ is an open map and the inverse $\phi_{I\tau}^{-1}\colon \phi_{I\tau}(X) \to X$ is continuous. Then (X, dist) and $(\phi_{I\tau}(X), \operatorname{dist}_H)$ are homeomorphic. The distance dist in X is the pull-back of dist_H by $\phi_{I\tau}$, so dist and dist are equivalent metrics in X. □

The following propositions deal with the following question. For y close to $\phi_{-t_0}x$ for some $t_0 > 0$, is it true that $\text{dist}(y, x) \geq \text{dist}(\phi_t y, x)$ for small and positive values of t? The following example shows that it is not always the case.

Example 7.2.4. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x\sin(1/x)$ and $f(0) = 0$. Consider the flow in $\mathbb{R}^2$ (with the Euclidean metric) defined as $\phi_t(x, f(x) + y) = (x + t, f(x + t) + y)$. It is easy to see that for $x = (0, 0)$ the function $t \mapsto \text{dist}(x, \phi_t(x))$ is not increasing on any interval $[0, \varepsilon]$. Therefore, the property mentioned above does not hold in this case. We have that this flow is conjugated with $\psi_t(x, y) = (x + t, y)$, so, one can think that the* problem *is with the metric.*

Assume that the flow has no singular points. We will show that our metric dist has not the *problem* shown in the previous example. To continue we need the following lemma. It is stated for the inverse flow, defined as $\phi_t^{-1} = \phi_{-t}$, because in the following proposition it will be used in this way.

Lemma 7.2.5. *If $\phi_s^{-1}x \neq x$ for all $x \in X$ and $s \in (0, 3\tau]$ then there is $\tilde\tau > 0$ such that for all $p \in X$, $\text{dist}(p, \phi_\theta^{-1}p) < \text{dist}(p, \phi_{\theta+2\tau}^{-1}p)$ for all $p \in X$ and $\theta \in [0, \tilde\tau]$.*

Proof. By contradiction assume that there is $\theta_n > 0$, $\theta_n \to 0$, and $p_n \to p_*$ such that $\text{dist}(p_n, \phi_{\theta_n}^{-1}p_n) \geq \text{dist}(p_n, \phi_{\theta_n+2\tau}^{-1}p_n)$ for all $n \geq 0$. Then, in the limit, we have the contradiction $\phi_{2\tau}^{-1}p_* = p_*$. □

Now we can prove the main result of this section. We assume that there are no periods smaller than 3τ.

Proposition 7.2.6. *If ϕ has no singular points then for all $t_0 \in (0, \tilde\tau]$ ($\tilde\tau$ is given in the previous lemma) there is $\delta > 0$ and $t_1 \in (0, t_0)$ such that if $\text{dist}(\phi_{t_o}y, x) < \delta$ and $0 \leq s \leq u \leq t_1$ then $\text{dist}(\phi_s y, x) \geq \text{dist}(\phi_u y, x)$.*

Proof. By contradiction assume that there is $t_0 \in (0, \tilde\tau]$, sequences $x_n, y_n \in X$ and $s_n, u_n \in \mathbb{R}$ such that $\phi_{t_0}y_n \to z$, $x_n \to z$, $0 \leq s_n \leq u_n \to 0$ and

$$\text{dist}(\phi_{s_n}y_n, x_n) < \text{dist}(\phi_{u_n}y_n, x_n) \tag{7.1}$$

for all $n \geq 0$. Inequality (7.1) means that there is $\varepsilon_n > 0$ such that

(a) $\phi_{I\tau}(\phi_{s_n}y_n) \subset B_{\varepsilon_n}(\phi_{I\tau}(x_n))$ and

(b) $\phi_{I\tau}(x_n) \subset B_{\varepsilon_n}(\phi_{I\tau}(\phi_{s_n}y_n))$

but

(c) $\phi_{I\tau}(\phi_{u_n}y_n) \not\subset B_{\varepsilon_n}(\phi_{I\tau}(x_n))$ or

(d) $\phi_{I\tau}(x_n) \not\subset B_{\varepsilon_n}(\phi_{I\tau}(\phi_{u_n}y_n))$.

In this paragraph we show that ε_n does not converge to 0. By (a) we have that there is $w_n \in I\tau$ such that

$$\mathrm{dist}(\phi_{-\tau+s_n} y_n, \phi_{w_n} x_n) < \varepsilon_n. \tag{7.2}$$

Taking a subsequence we can assume that $w_n \to w_* \in I\tau$. Taking limit in the inequality (7.2) and supposing that $\varepsilon_n \to 0$ we have that $\phi_{-\tau-t_0} z = \phi_{w_*} z$. This is a contradiction because $z = \phi_{\tau+t_0+w_*} z$ and $|\tau + t_0 + w_*| < 3\tau$. So, taking a subsequence of ε_n, we assume that $\varepsilon_n \to \varepsilon_* > 0$.

Assume that (c) holds. It implies that there is $v_n \in I\tau$ such that for all $t \in I\tau$

$$\mathrm{dist}(\phi_{v_n+u_n} y_n, \phi_t x_n) \geq \varepsilon_n. \tag{7.3}$$

Now we show that $v_n \to \tau$. By (a) we have that for all $s \in I\tau$, there is $t \in I\tau$ such that

$$\mathrm{dist}(\phi_{s+s_n} y_n, \phi_t x_n) < \varepsilon_n. \tag{7.4}$$

Using the inequalities (7.3) and (7.4) we have that $s + s_n \neq v_n + u_n$ for all $s \in I\tau$. But $v_n \in I\tau$, so $v_n \in (\tau - (u_n - s_n), \tau]$. Then $v_n \to \tau$.

Now, taking limit in the inequality (7.3) we have that $\mathrm{dist}(\phi_{\tau-t_0} z, \phi_t z) \geq \varepsilon_*$ for all $t \in I\tau$. So we can put $t = \tau - t_0$ and $\mathrm{dist}(z, z) \geq \varepsilon_* > 0$ which is a contradiction. Then (c) cannot hold.

Now assume that (d) is true. Condition (d) means that there is $v_n \in I\tau$ such that for all $t \in I\tau$ we have

$$\mathrm{dist}(\phi_{v_n} x_n, \phi_{t+u_n} y_n) \geq \varepsilon_n. \tag{7.5}$$

By (b) we have that there is $w_n \in I\tau$ such that

$$\mathrm{dist}(\phi_{v_n} x_n, \phi_{s_n+w_n} y_n) < \varepsilon_n. \tag{7.6}$$

We will show that $w_n \to -\tau$. By (7.5) and (7.6) we have that $s_n + w_n \neq t + u_n$ for all $t \in I\tau$. Then $w_n \notin [-\tau + u_n - s_n, \tau + u_n - s_n]$ but $w_n \in I\tau$. Therefore $w_n \in [-\tau, -\tau + u_n - s_n)$ and $w_n \to -\tau$.

Assuming that $v_n \to v_* \in I\tau$ and taking limit in (7.5) we have that

$$\mathrm{dist}(\phi_{v_*} z, \phi_{t-t_0} z) \geq \varepsilon_* \tag{7.7}$$

for all $t \in I\tau$. Also, taking limit in (7.6) we have

$$\mathrm{dist}(\phi_{v_*} z, \phi_{-\tau-t_o} z) \leq \varepsilon_*. \tag{7.8}$$

By (7.7) and the fact that $\varepsilon_* > 0$ we have that $v_* \neq t - t_0$ for all $t \in I\tau$. Then $v_* \in (\tau - t_0, \tau]$. If $t = \tau$ in inequality (7.7) we have that $\mathrm{dist}(\phi_{v_*} z, \phi_{\tau-t_0} z) \geq \varepsilon_*$. This and inequality (7.8) contradict Lemma 7.2.5, with $\theta = v_* - (\tau - t_0)$ and $p = \phi_{v_*} z$, because $t_0 \in (0, \tilde{\tau}]$. □

Proposition 7.2.7. *For all* $t_2 \in (0, \tilde{\tau}]$ *there are* $\delta > 0$ *and* $t_1 > 0$ *such that if* $\operatorname{dist}(\phi_t x, y) < \delta$ *or* $\operatorname{dist}(x, \phi_{-t} y) < \delta$ *for some* $t \in [t_2, \tilde{\tau}]$ *and* $0 \leq s \leq u \leq t_1$ *then* $\operatorname{dist}(\phi_s y, x) \geq \operatorname{dist}(\phi_u y, x)$.

Proof. It follows by Proposition 7.2.6 and the compactness of the interval $[t_2, \tilde{\tau}]$. □

7.2.2 Expansive flows

In this section we present the definition of expansive flow and some useful equivalences. We state them for positive expansiveness but they have their counterpart for expansive flows. We consider flows without singular points. In Section 7.2.5 we consider the singular case.

Let $\mathcal{H}^+$ be the set of all increasing homeomorphisms $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$. Such maps are called *reparameterizations*.

Definition 7.2.8. A continuous flow ϕ on a compact metric space X is *positive expansive* if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $\operatorname{dist}(\phi_{h(t)} x, \phi_t y) < \delta$ for all $t \geq 0$, with $x, y \in X$ and $h \in \mathcal{H}^+$, then $y \in \phi_{I\varepsilon} x$.

Recall that $y \in \phi_{I\varepsilon} x$ if and only if there is $t \in I\varepsilon = [-\varepsilon, \varepsilon]$ such that $y = \phi_t x$. This is the *positive* adaptation of the definition given by R. Bowen and P. Walters in [17]. Now we present an equivalent definition. Consider $\mathcal{H}$ as the set of non-decreasing, surjective and continuous maps $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$. By *non-decreasing* we mean: if $s < t$ then $h(s) \leq h(t)$. The idea is to allow a point to stop the clock for a while (recall that in [80] reparameterizations are called *clocks*). The maps of $\mathcal{H}$ will be called *reparameterizations with rests*.

Define the set of pairs of reparameterizations with rests

$$\mathcal{H}^2 = \{g = (h_1, h_2) : h_1, h_2 \in \mathcal{H}\}$$

and extend the action of ϕ to $X \times X$ as $\phi_t(x, y) = (\phi_t x, \phi_t y)$. Also we define

$$\phi_{g(t)}(x, y) = (\phi_{h_1(t)} x, \phi_{h_2(t)} y)$$

for $g = (h_1, h_2) \in \mathcal{H}^2$. We now consider the Fréchet distance defined by

$$\operatorname{dist}_F(x, y) = \inf_{g \in \mathcal{H}^2} \sup_{t \geq 0} \operatorname{dist}(\phi_{g(t)}(x, y)).$$

This distance was introduced in [33] in the beginning of the Theory of metric spaces. It was first defined for compact curves but, as noticed in [80], it can be extended to non-compact trajectories.

Proposition 7.2.9. *A flow* ϕ *is positive expansive if and only if for all* $\varepsilon > 0$ *there is* $\delta > 0$ *such that if* $\operatorname{dist}_F(x, y) < \delta$ *then* x *and* y *are in an* ε*-orbit segment.*

Proof. The converse follows because $id_{\mathbb{R}} \in \mathcal{H}^+ \subset \mathcal{H}$. The direct part is a consequence of the following lemma. □

Lemma 7.2.10. *For all $\delta > 0$ there is $\delta' > 0$ such that if $\operatorname{dist}_F(x, y) < \delta'$ then there is $h \in \mathcal{H}^+$ such that $\operatorname{dist}(\phi_{h(t)}x, \phi_t y) < \delta$ for all $t \geq 0$.*

Proof. Given $\delta > 0$ consider $\delta' \in (0, \delta)$ and $\gamma > 0$ such that $\operatorname{dist}(x, \phi_t x) < (\delta - \delta')/2$ for all $x \in X$ and for all $t \in (-\gamma, \gamma)$. Suppose that $\operatorname{dist}_F(x, y) < \delta'$ for some $x, y \in X$. Then, there is $g = (h_x, h_y) \in \mathcal{H}^2$ such that $\operatorname{dist}(\Phi_{g(t)}(x, y)) < \delta'$ for all $t \geq 0$. Take two increasing sequences s_n and t_n such that $h_x(t_n) = h_y(s_n) = n\gamma$ for all $n \geq 1$, starting with $s_0 = t_0 = 0$. Then define $h_1(t_n) = h_2(s_n) = n\gamma$ and extend piecewise linearly. In this way we have that $|h_1(t) - h_x(t)|, |h_2(t) - h_y(t)| < \gamma$ for all $t \geq 0$. Then by the triangular inequality it follows that $h = h_1 \circ h_2^{-1}$ works. □

Consider the set $T_\varepsilon(x, y) \subset \mathbb{R}^2$ of pairs of positive numbers (t_x, t_y) such that there are $g \in \mathcal{H}^2$ and $s > 0$ such that $\operatorname{dist}(\phi_{g(t)}(x, y)) \leq \varepsilon$ for all $t \in [0, s]$ and $g(s) = (t_x, t_y)$. In $\mathbb{R}^2$ we consider the norm $\|(a, b)\| = |a| + |b|$ (the properties of this specific norm will be used in the next section).

Remark 7.2.11. *If $T_\delta(x, y)$ is not bounded then $\pi_1 T_\delta(x, y)$ and $\pi_2 T_\delta(x, y)$ are not bounded, where $\pi_i(x_1, x_2) = x_i$, $i = 1, 2$, are the canonical projections of $\mathbb{R}^2$.*

Lemma 7.2.12. *For all $T, \delta' > 0$ there is $\delta > 0$ such that if $\operatorname{dist}(\phi_{g(t)}(x, y)) < \delta$ for all $t \in [0, T]$ and some $g \in \mathcal{H}^2$ then there is $h \in \mathcal{H}^+$ such that $\operatorname{dist}(\phi_{h(t)}x, \phi_t y) < \delta'$ for all $t \in [0, h(T)]$.*

Proof. Use the same technique of Lemma 7.2.10. □

If $\operatorname{dist}_F(x, y) < \varepsilon$ then $T_\varepsilon(x, y)$ is not bounded, as can be seen from the definitions. The following proposition is a kind of converse. Its proof is based on the proof of Lemma 9 in [124].

Proposition 7.2.13. *For all $\varepsilon > 0$ there is $\delta > 0$ such that if $T_\delta(x, y)$ is not bounded then $\operatorname{dist}_F(x, y) < \varepsilon$.*

Proof. For $\varepsilon > 0$ given consider $\gamma > 0$ such that

$$\text{if } \operatorname{dist}(x, y) < \varepsilon/2 \text{ and } |t| < \gamma \text{ then } \operatorname{dist}(\phi_t x, y) < \varepsilon. \tag{7.9}$$

Take $\delta' \in (0, \varepsilon/2)$ such that

$$\text{if } \operatorname{dist}(x, y) < \delta' \text{ then } \operatorname{dist}(\phi_{\pm\gamma} x, y) > \delta'. \tag{7.10}$$

Finally, pick $\delta > 0$ from Lemma 7.2.12 associated to δ'. We will show that this value of δ works. Suppose that for some $x, y \in X$ we have that $T_\delta(x, y)$ is not bounded. So, for all $n \geq 1$ there are $h'_x, h'_y \in \mathcal{H}$ and $T > 0$ such that

$$\operatorname{dist}(\phi_{h'_x(t)}x, \phi_{h'_y(t)}y) < \delta$$

for all $t \in [0, T]$ and $h'_y(T) = n$. Then by Lemma 7.2.12 there is $h^n_x \in \mathcal{H}$ such that

$$\operatorname{dist}(\phi_{h^n_x(t)}x, \phi_t y) < \delta'$$

for all $t \in [0, n]$. Eventually taking a subsequence we can suppose that there is an increasing sequence $w_n \to \infty$ such that $h^n_x(w_n) = n\gamma$ and

$$\operatorname{dist}(\phi_{h^n_x(t)}x, \phi_t y) < \delta'$$

for all $t \in [0, w_n]$. We will define $h \in \mathcal{H}$ such that

$$\operatorname{dist}(\phi_{h(t)}x, \phi_t y) < \varepsilon$$

for all $t \geq 0$. Define $h(w_n) = h^n_x(w_n) = n\gamma$ for all $n \geq 0$. For $t \in [0, w_1]$ define $h(t) = h^1_x(t)$. Now consider $t \in (w_{n-1}, w_n)$. To define $h(t)$ we consider two cases.

1. If $h^{n-1}_x(w_{n-1}) \leq h^n_x(w_{n-1})$ then

   $$h(w_{n-1}) = h^{n-1}_x(w_{n-1})$$

   and extend linearly for $t \in (w_{n-1}, w_n)$.
2. If $h^{n-1}_x(w_{n-1}) > h^n_x(w_{n-1})$ consider $z \in (w_{n-1}, w_n)$ such that $h^n_x(z) = (n-1)\gamma$. Define $h(t) = (n-1)\gamma$ for all $t \in [w_{n-1}, z]$ and extend linearly for $t \in [z, w_n]$.

By condition (7.10) we have that $|h(t) - h^n_x(t)| \leq \gamma$ for all $t \in [w_{n-1}, w_n]$ and $n \geq 1$. Then, since $\operatorname{dist}(\phi_{h^n_x(t)}x, \phi_t y) < \delta' < \varepsilon/2$, we have by condition (7.9) that

$$\operatorname{dist}(\phi_{h(t)}x, \phi_t y) < \varepsilon$$

for all $t \geq 0$ and the proof ends. □

Here is another characterization of expansiveness that will be useful.

Proposition 7.2.14. *A flow ϕ is positive expansive if and only if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $T_\delta(x, y)$ is not bounded then x and y are in an ε-orbit segment.*

Proof. Suppose that ϕ is positive expansive. Consider $\varepsilon > 0$ given. By Proposition 7.2.9 there is δ' such that if $\operatorname{dist}_F(x, y) < \delta'$ then they are in a ε-orbit segment. Now take from Proposition 7.2.13 a positive δ such that if $T_\delta(x, y)$ is not bounded then $\operatorname{dist}_F(x, y) < \delta'$. This finishes the direct part.

The converse follows because if $\operatorname{dist}_F(x, y) < \delta$ then $T_\delta(x, y)$ is not bounded. □

7.2.3 Stability

In this section we assume that the flow has no singular points. We introduce the concept of Lyapunov stability allowing reparameterizations of the trajectories. The stability properties of positive expansive flows are stated. We assume that the metric of the space is dist, defined in Section 7.2.1.

We start defining Lyapunov stability according to the Fréchet distance as was done in [78, 100].

Definition 7.2.15. We say that x is *stable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $\operatorname{dist}(x, y) < \delta$ then $\operatorname{dist}_F(x, y) < \varepsilon$, i.e. there is a pair of reparameterizations with rests $g \in \mathcal{H}^2$ such that $\operatorname{dist}(\phi_{g(t)}(x, y)) < \varepsilon$ for all $t \geq 0$.

Remark 7.2.16. *By Lemma 7.2.10 we have that x is stable if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $\operatorname{dist}(x, y) < \delta$ then there is a reparameterization $h \in \mathcal{H}^+$ such that $\operatorname{dist}(\phi_t x, \phi_{h(t)} y) < \varepsilon$ for all $t \geq 0$.*

Definition 7.2.17. We say that (T_x, T_y) in the closure of $T_\varepsilon(x, y)$ is a *maximal pair of times* for (ε, x, y) if for all $(t_x, t_y) \in T_\varepsilon(x, y)$ we have that $\|(T_x, T_y)\| \geq \|(t_x, t_y)\|$ for the sum norm in $\mathbb{R}^2$.

In the following result we use the properties of dist (Proposition 7.2.7). For this we will consider the positive number $\tilde{\tau}$ given in Lemma 7.2.5 and the interval $I\tilde{\tau} = [-\tilde{\tau}, \tilde{\tau}]$. As usual, we define the distance between a point $a \in X$ and a set $A \subset X$ as $\operatorname{dist}(a, A) = \inf\{\operatorname{dist}(a, x) : x \in A\}$.

Proposition 7.2.18. *For all $\varepsilon > 0$ there is $\sigma > 0$ such that if (T_x, T_y) is a maximal pair of times for (ε, x, y) then*

$$\operatorname{dist}(\phi_{T_x} x, \phi_{I\tilde{\tau}}(\phi_{T_y} y)) > \sigma \text{ and } \operatorname{dist}(\phi_{T_y} y, \phi_{I\tilde{\tau}}(\phi_{T_x} x)) > \sigma.$$

Proof. Given $\varepsilon > 0$ consider $t_2 > 0$ such that $\phi_{[-t_2, t_2]} x \subset B_\varepsilon(x)$ for all $x \in X$. For this value of t_2 take $\delta > 0$ and $t_1 > 0$ from Proposition 7.2.7. Consider $\sigma \in (0, \delta)$ such that

$$\text{if } y \notin B_\varepsilon(x) \text{ then } \operatorname{dist}(\phi_{[-t_2, t_2]} x, y) > \sigma. \tag{7.11}$$

Notice that $\operatorname{dist}(\phi_{T_x} x, \phi_{T_y} y) = \varepsilon$. By contradiction assume that

$$\operatorname{dist}(\phi_{T_y} y, \phi_{I\tilde{\tau}}(\phi_{T_x} x)) \leq \sigma,$$

being the other case symmetric. By condition (7.11) there is $t_0 \in [-\tilde{\tau}, -t_2] \cup [t_2, \tilde{\tau}]$ such that

$$\operatorname{dist}(\phi_{T_y} y, \phi_{t_0} \phi_{T_x} x) \leq \sigma.$$

Suppose that $t_0 \in [t_2, \tilde{\tau}]$ (the other case is similar). Now take $g \in \mathcal{H}^2$, $(T'_x, T'_y) \in \mathbb{R}^2$ and $s > 0$ such that $\operatorname{dist}(\phi_{g(t)}(x, y)) < \varepsilon$ for all $t \in [0, s]$,

$$\|(T'_x, T'_y) - (T_x, T_y)\| < t_1 \tag{7.12}$$

and $g(s) = (T'_x, T'_y)$. We define $\hat{g} \in \mathcal{H}^2$ as

$$\hat{g}(t) = \begin{cases} g(t) & \text{for all } t \leq s, \\ g(s) + (t - s, 0) & \text{if } t \in [s, s + t_1], \\ g(s) + (t - s, t - s - t_1) & \text{if } t \geq s + t_1. \end{cases}$$

So, for $t \in [s, s + t_1]$ we have, by Proposition 7.2.7, that

$$\operatorname{dist}(\phi_{\hat{g}(t)}(x, y)) \leq \operatorname{dist}(\phi_{\hat{g}(s)}(x, y)) < \varepsilon.$$

Then $g(s + t_1) = (T'_x + t_1, T'_y) \in T_\varepsilon(x, y)$ and by inequality (7.12) we have that $\|g(s + t_1)\| > \|(T_x, T_y)\|$ contradicting the maximality of (T_x, T_y). □

Given $\varepsilon > 0$ and $x, y \in X$ we consider the following set of pairs of reparameterizations with rests

$$\mathcal{H}^2_\varepsilon(x, y) = \{g \in \mathcal{H}^2 : \operatorname{dist}(\phi^{-1}_{g(t)}(x, y)) < \varepsilon \text{ for all } t \geq 0\}.$$

The following result says that if two points are close enough then $\mathcal{H}^2_\varepsilon(x, y)$ is not empty if ϕ is positive expansive without singular points. Notice that positive expansiveness does not depend on the metric (defining the same topology).

Lemma 7.2.19. *If ϕ is positive expansive then every point is stable for ϕ^{-1} with uniform δ.*

Proof. By Proposition 7.2.14 there is an expansive constant $\varepsilon' > 0$ such that if $T_{\varepsilon'}(x, y)$ is not bounded then $y \in \phi_{I_{\tilde{\tau}}} x$. By contradiction assume that there are $\varepsilon \in (0, \varepsilon')$ and two sequences x_j, y_j such that $\operatorname{dist}(x_j, y_j) \to 0$ as $j \to \infty$ and $T_\varepsilon(x_j, y_j)$ (defined for ϕ^{-1}) is bounded for all $j \in \mathbb{N}$. For each j consider (T_{x_j}, T_{y_j}) a maximal pair of times for (ε, x_j, y_j) associated to ϕ^{-1}. By the continuity of the flow we have that $T_{x_j}, T_{y_j} \to \infty$ as $j \to \infty$. Eventually taking subsequences, we can assume that $\phi_{T_{x_j}} x_j \to x_*$ and $\phi_{T_{y_j}} y_j \to y_*$. By Proposition 7.2.18 we have that x_* and y_* are not in a $\tilde{\tau}$-orbit segment. Also, for every $T > 0$ we have that there are $g \in \mathcal{H}^2$ and $s > 0$ such that $\operatorname{dist}(\phi_{g(t)}(x_*, y_*)) < \varepsilon'$ for all $t \in [0, s]$ and $\|g(s)\| \geq T$. So, $T_{\varepsilon'}(x_*, y_*)$ is not bounded and it contradicts the positive expansiveness of the flow (as stated in Proposition 7.2.14) because x_* and y_* are not in a $\tilde{\tau}$-orbit segment. □

The following lemma states the uniform asymptotic stability for $t \to -\infty$.

Lemma 7.2.20. *If ϕ is positive expansive then for all $\varepsilon > 0$ there is $\delta > 0$ such that for all $\sigma > 0$ there is $T > 0$ such that if $\operatorname{dist}(x, y) < \delta$ then there is $g \in \mathcal{H}^2_\varepsilon(x, y)$ such that $\operatorname{dist}(\phi^{-1}_{g(t)}(x, y)) < \sigma$ if $\|g(t)\| \geq T$.*

Proof. Given $\varepsilon > 0$ smaller than an expansive constant, consider $\delta > 0$ from Lemma 7.2.19. By contradiction we will show that this value of δ works. So, suppose that there are $\sigma > 0$, $T_n \to \infty$ and $x_n, y_n \in X$ such that $\mathrm{dist}(x_n, y_n) < \delta$ and

$$\begin{gathered}\text{for all } g \in \mathcal{H}^2_\varepsilon(x_n, y_n) \text{ there is } t \geq 0 \text{ such that}\\ \|g(t)\| \geq T_n \text{ and } \mathrm{dist}(\phi^{-1}_{g(t)}(x_n, y_n)) \geq \sigma.\end{gathered} \tag{7.13}$$

Again by Lemma 7.2.19 there is δ' such that

$$\text{if } \mathrm{dist}(u, v) < \delta' \text{ then } \mathcal{H}^2_\sigma(u, v) \text{ is not empty.} \tag{7.14}$$

For each n take $g_n \in \mathcal{H}^2_\varepsilon(x_n, y_n)$ and consider t_n such that $\|g_n(t_n)\| = T_n - \tilde{\tau}$. Let $(u_n, v_n) = \phi^{-1}_{g_n(t_n)}(x_n, y_n)$. By conditions (7.13) and (7.14) there is δ'' such that $\mathrm{dist}(\phi_t u_n, v_n) \geq \delta''$ and $\mathrm{dist}(u_n, \phi_t v_n) \geq \delta''$ if $|t| \leq \tilde{\tau}$. So, limit points of u_n and v_n are not in a $\tilde{\tau}$-orbit segment and contradict positive expansiveness. □

7.2.4 Positive expansiveness

In this section we prove the main result of this chapter. We consider flows without singular points. First we show that positive expansive flows have periodic orbits. The idea to find such trajectories is to show that there is a compact invariant set that is a suspension and apply the result for positive expansive homeomorphisms.

Lemma 7.2.21. *Every positive expansive flow has at least one periodic orbit.*

Proof. Consider $\varepsilon' > 0$ such that for all $y \in X$

$$\text{if } (\tilde{h}, \tilde{h}') \in \mathcal{H}^2_{2\varepsilon'}(y, y) \text{ then } |\tilde{h}(t) - \tilde{h}'(t)| < \tilde{\tau}/2 \text{ for all } t \geq 0. \tag{7.15}$$

Take a recurrent point x and $t_n \to +\infty$ such that $\phi^{-1}_{t_n}(x) \to x$. For any $\varepsilon \in (0, \varepsilon')$ consider $\delta > 0$ from Lemma 7.2.20. Let $S \subset B_\delta(x)$ be a compact local cross section of time $\tilde{\tau}$, $x \in S$, and consider the flow box $U = \phi_{[-\tilde{\tau}, \tilde{\tau}]}(S)$. Consider $r > 0$ such that

$$\phi_{[-\tilde{\tau}/2, \tilde{\tau}/2]} B_r(x) \subset U. \tag{7.16}$$

For $\sigma = r/2$ in Lemma 7.2.20 take the corresponding $T > 0$. Let $N > 0$ be such that $\mathrm{dist}(\phi^{-1}_{t_N} x, x) < r/2$ and $t_N > T$. By Lemma 7.2.20, for all $y \in S$ ($S \subset B_\delta(x)$) there is $g \in \mathcal{H}^2_\varepsilon(x, y)$ such that:

$$\mathrm{dist}(\phi^{-1}_{g(t)}(x, y)) < \sigma = r/2$$

if $\|g(t)\| \geq T$. If $g = (h_x, h_y)$ there is $s \geq 0$ such that $h_x(s) = t_N$. Then $\|g(s)\| \geq T$ and $\phi^{-1}_{h_y(s)} y \in B_r(x) \subset U$. Consider $\pi \colon U \to S$ the projection on the flow box. Let $f \colon S \to S$ be defined by

$$f(y) = \pi(\phi^{-1}_{h_2(s)} y)$$

if $s \geq 0$ and $g = (h_1, h_2) \in \mathcal{H}^2_\varepsilon(x, y)$ satisfy:

1. $h_1(s) = t_N$ and
2. $\phi^{-1}_{h_2(s)} y \in B_r(x)$.

We have shown that for all $y \in S$ there are s and g satisfying this conditions.

In this paragraph we will show that f is well defined, i.e. it does not depend on g and s. Consider $s, s' \geq 0$ and $g = (h_1, h_2), g' = (h'_1, h'_2) \in \mathcal{H}^2_\varepsilon(x, y)$ satisfying both items above. Recall that $\varepsilon' > \varepsilon$ and consider two increasing reparameterizations $\hat{h}_1$ and $\hat{h}'_1$ such that

- $\mathrm{dist}(\phi^{-1}_{\hat{h}_1(t)} x, \phi^{-1}_{h_2(t)} y) < \varepsilon'$ for all $t \geq 0$,
- $\mathrm{dist}(\phi^{-1}_{\hat{h}'_1(t)} x, \phi^{-1}_{h'_2(t)} y) < \varepsilon'$ for all $t \geq 0$ and
- $\hat{h}_1(s) = t_N = \hat{h}'_1(s')$.

So, if we define $(\tilde{h}, \tilde{h}') = (h_2 \circ \hat{h}_1^{-1}, h'_2 \circ \hat{h}_1'^{-1})$ we have that

- $\mathrm{dist}(\phi^{-1}_t x, \phi^{-1}_{\tilde{h}(t)} y) < \varepsilon'$ for all $t \geq 0$,
- $\mathrm{dist}(\phi^{-1}_t x, \phi^{-1}_{\tilde{h}'(t)} y) < \varepsilon'$ for all $t \geq 0$,
- $h_2(s) = \tilde{h}(t_N)$ and $h'_2(s') = \tilde{h}'(t_N)$.

and by the triangular inequality

$$\mathrm{dist}(\phi^{-1}_{\tilde{h}(t)} y, \phi^{-1}_{\tilde{h}'(t)} y) < 2\varepsilon'$$

for all $t \geq 0$. Then by condition (7.15) we have that

$$|h_2(s) - h'_2(s)| = |\tilde{h}(t_N) - \tilde{h}'(t_N)| < \tilde{\tau}/2.$$

This inequality joint with Eq. (7.16) and the fact that $\phi^{-1}_{h_2(s)} y, \phi^{-1}_{h'_2(s')} y \in B_r(x)$ implies that the points $\phi^{-1}_{h_2(s)} y$ and $\phi^{-1}_{h'_2(s')} y$ are in the same orbit segment contained in the flow box U. So, they have the same projection in the local cross section S and f is well defined.

Now we will show that f is continuous. Given $y \in S$ consider $s \geq 0$ and $g = (h_1, h_2) \in \mathcal{H}^2_\varepsilon(x, y)$ satisfying the definition of $f(y)$. Consider $\rho > 0$ such that for all $y' \in B_\rho(y) \cap S$ we have that $\phi^{-1}_{h_2(s)} y' \in B_r(x)$. Then the continuity of f follows by the continuity of the flow ϕ and the continuity of the projection π.

Now one can restrict f to the compact invariant set

$$K = \cap_{n \geq 0} f^n(S)$$

and notice that f is a negative expansive homeomorphisms on K because ϕ is positive expansive in $\phi_\mathbb{R}(K)$. We conclude that K is finite and f has periodic points. So ϕ has periodic orbits. □

Theorem 7.2.22. *If ϕ is a positive expansive flow without singular points then X is the union of a finite number of periodic orbits.*

Proof. First we show that every orbit is periodic. By contradiction assume that there is a point x whose orbit is not compact. By Lemma 7.2.21 there is a periodic orbit contained in $\omega(x)$. But it contradicts Lemma 7.2.20. Again by Lemma 7.2.20 there is just a finite number of periodic orbits and the proof ends. □

7.2.5 Positive expansive singular flows

Now we consider positive expansive flows with singular points. A change in the definition is needed because singularities are isolated points of the space if the flow is expansive according to Definition 7.2.8 (even if one considers expansiveness instead of positive expansiveness). So, for singular flows we consider the following definition.

Definition 7.2.23. A continuous flow ϕ in a compact metric space X is *positive expansive* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\operatorname{dist}(\phi_{h(t)}x, \phi_t y) < \delta$ for all $t \geq 0$, with $x, y \in X$ and $h \in \mathcal{H}^+$, then x and y are in an orbit segment of diameter smaller than ε.

This is the *positive* adaptation of the definition given in [5] for expansive flows with singular points. Definitions 7.2.8 and 7.2.23 coincide if the flow has no singular points.

Theorem 7.2.24. *If ϕ is a positive expansive flow with singular points then X is the union of finite periodic orbits and singularities.*

Proof. Let $\varepsilon > 0$ be an expansive constant. We will show that singularities are stable for ϕ^{-1}. By contradiction assume there are $x_n \to p$, $x_n \neq p$, p a singular point, and for all $n \in \mathbb{N}$ there is $t_n \geq 0$ such that $\operatorname{dist}(\phi^{-1}_{t_n} x_n, p) = \varepsilon$. If $y_n = \phi^{-1}_{t_n} x_n$ converges to q, then $q \neq p$ and $\phi_t q \to p$ as $t \to \infty$. So, p and q contradict the positive expansiveness of the flow. Therefore there is $\delta > 0$ such that if $\operatorname{dist}(x, p) < \delta$ then $\phi^{-1}_t x \in B_\varepsilon(p)$ for all $t \geq 0$.

We will show that $B_\delta(p) = \{p\}$. By contradiction suppose that $\operatorname{dist}(x, p) \in (0, \delta)$ for some x. By hypothesis there is $t > 0$ such that $\phi_t x \notin B_\varepsilon(p)$. So x is not periodic. By the stability of singularities there is no singular point in $\omega(x)$. Then $\omega(x)$ is positive expansive, connected and free of singularities. By Theorem 7.2.22 it is a periodic orbit. But this contradicts the stability of periodic orbits, i.e. Lemma 7.2.19. So, singular points are isolated points of X and the proof is reduced to Theorem 7.2.22. □

Chapter 8

Robust expansiveness

In this section we study the persistence of expansiveness under perturbations of the velocity field in the C^1-topology. On surfaces there are no robust geometric expansive flows because small C^1-perturbations gives rise to periodic orbits (see [103]) and this is an obstruction to geometric expansiveness (see [5]). As a corollary we have that there are no robust geometric expansive flows on three dimensional manifolds with non-empty boundary.

On surfaces we will consider robust kinematic expansiveness in the conservative framework. On manifolds of dimension greater than two we will prove that robust kinematic expansiveness is equivalent with geometric expansiveness.

8.1 Positive expansiveness in the annulus

Let $A \subset \mathbb{R}^2$ be the annulus bounded by two simple closed C^1 curves as in Figure 8.1. Denote by $\mathbb{X}^1_\mu(A)$ the vector space of C^1 vector fields X defined in A such that

1. $\mathrm{div}(X) = 0$ and
2. X is parallel to ∂A in ∂A.

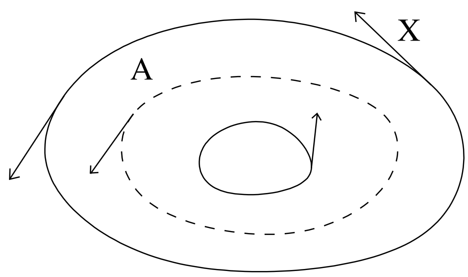

Figure 8.1: A vector field in the annulus tangent to the boundary.

Definition 8.1.1. We say that $X \in \mathbb{X}^1_\mu(A)$ is *robustly positive kinematic expansive* if there is a C^1-neighborhood of X in $X^1_\mu(A)$ such that every vector field in this neighborhood gives rise to a positive kinematic expansive flow.

If $X = (a, b)$ denote $X^\perp = (-b, a)$.

Theorem 8.1.2. *Let $X \in \mathbb{X}^1_\mu(A)$ be a non-vanishing vector field and define $Z = X^\perp/\|X\|^2$. If*

$$\operatorname{div}(Z) \neq 0 \tag{8.1}$$

on every point of A then X is robustly positive kinematic expansive.

Proof. Let us first recall that if $\operatorname{div}(X) = 0$ then there are no wandering points and since X has no singularities, we have that every orbit is periodic because no other kind of recurrence is possible in the annulus in our hypothesis. It implies that the flow is a suspension of the identity in a global cross section. Then, in order to prove kinematic expansiveness it is enough to prove that different periodic orbits have different periods.

Let γ be a periodic orbit of X. The period of γ, denoted by $T(\gamma)$, can be calculated as follows:

$$T(\gamma) = \int_\gamma \frac{1}{\|X\|}\, d\gamma = \int_\gamma Z \cdot n \, d\gamma,$$

where n is the normal vector of γ in the direction of Z. That is, the period of γ is the flow of Z through γ. Now consider two periodic orbits γ_1 and γ_2 bounding a region R as in Figure 8.2.

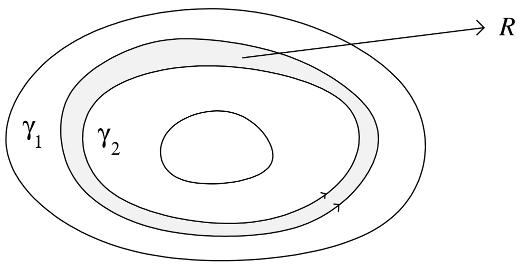

Figure 8.2: Region R bounded by two periodic orbits.

Applying Green's Theorem we have that

$$\int_{\gamma_2} Z \cdot n \, d\gamma_2 - \int_{\gamma_1} Z \cdot n \, d\gamma_1 = \iint_R \operatorname{div}(Z)\, dx\, dy.$$

And then

$$T(\gamma_2) - T(\gamma_1) = \iint_R \operatorname{div}(Z)\, dx\, dy \neq 0.$$

Then we have proved that different periodic orbits have different periods and then X is kinematic expansive. It only rests to notice that condition (8.1) cannot be lost by a C^1 small perturbation of X. □

Example 8.1.3. *Given $r_1 > 0$ and $r_2 > r_1$ consider the annulus $A \subset \mathbb{R}^2$ given by $r_1^2 \leq x^2+y^2 \leq r_2^2$. Given a smooth non-vanishing function $f\colon \mathbb{R} \to \mathbb{R}$ define $X_f(x,y) = f(r^2)(y,-x)$, where $r^2 = x^2+y^2$. In this case*

$$Z = \frac{1}{r^2 f(r^2)}(x,y)$$

and

$$\operatorname{div}(Z) = -\frac{2f'}{f^2}.$$

Therefore, X_f is robust kinematic expansive in A if $f' \neq 0$ in $[r_1, r_2]$.

8.2 Robust expansiveness on manifolds

Let X be a C^1 vector field of a closed manifold M of dimension $n \geq 3$. Assume that M is endowed with a smooth structure and a smooth Riemannian metric. In this section we also assume that X has no singularities.

Definition 8.2.1. We say that X is *C^1-robust kinematic (or geometric) expansive* if every vector field in a suitable C^1-neighborhood of X is kinematic (or geometric) expansive.

Theorem 8.2.2. *Every C^1-robust kinematic expansive vector field without singularities on a closed smooth manifold is geometric expansive.*

Proof. Consider X a C^1-robust kinematic expansive vector field. Let us start proving that periodic orbits of X are hyperbolic. Now, with standard perturbation techniques (as, for example, in the proof of Proposition 1 of [88]), it can be proved that if a periodic orbit is not hyperbolic then there is a C^1-close vector field Y with an invariant annulus A filled with periodic orbits of Y. This gives a contradiction with geometric expansiveness, but in our case we have to give more arguments. Consider a new perturbation Z such that A is Z-invariant but with at least one non-periodic orbit. This easily contradicts Proposition 6.3.8.

Therefore, we have proved that every periodic orbit of every vector field in a suitable neighborhood of X is hyperbolic. A vector field with this property is usually called as *star flow.* In [36] it is proved (see Theorem A) that non-singular star flows satisfy Axiom A, i.e., periodic orbits are dense in $\Omega(X)$ and $\Omega(X)$ is hyperbolic.

Now we prove the quasi-transversality condition, that is:

$$T_xW^s(x) \cap T_xW^u(x) = \{0_x\} \tag{8.2}$$

for all $x \in M$, where $W^s(x)$ is the stable manifold and $W^u(x)$ is the unstable manifold of x defined as usual. For $x \in \Omega(X)$ the quasi-transversality condition holds because $\Omega(X)$ is hyperbolic. Consider $x \notin \Omega(X)$ and, arguing by contradiction, assume that (8.2) does not hold. With a C^1-perturbation Y of X we can also assume that x is in the stable set of a periodic orbit γ_1 and also in the unstable set of a periodic orbit γ_2. With another perturbation Z we can suppose that the intersection of the stable manifold of γ_1 with the unstable manifold of γ_2 and a local cross section through x contains an arc l containing x. Now, applying the Invariant Manifold Theorem for hyperbolic periodic orbits, it is easy to see that for all $\delta > 0$ there is a sub-arc $j \subset l$ such that $\operatorname{diam}(\phi_t(j)) < \delta$ for all $t \in \mathbb{R}$. Here ϕ denotes the flow of the vector field Z. This contradicts kinematic expansiveness.

Since X satisfies Axiom A and the quasi-transversality condition, we can apply the results of [88] to conclude that X is in fact robust geometric expansive. $\square$

Appendix A

Quasi-metric interpolation

In this section we follow [35]. A *quasi-metric* is a map $\rho\colon X \times X \to \mathbb{R}$ such that

1. $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$,
2. $\rho(x, y) = \rho(y, x)$ and
3. $\rho(x, y) \leq 2 \max\{\rho(x, z), \rho(z, y)\}$ (generalized triangle property)

for all $x, y, z \in X$.

A quasi-metric induces a topology in X as a metric does. Given $a, b \in X$ define

$$D(a, b) = \inf\{\rho(a, x_1) + \rho(x_1, x_2) + \rho(x_2, x_3) + \cdots + \rho(x_{n-1}, x_n) + \rho(x_n, b)\} \tag{A.1}$$

with $x_1, x_2, \ldots, x_n \in X$ and $n \in \mathbb{N}$. It is called the *interpolation* of the quasi-metric. We will show that D is a metric defining the same topology as ρ. Notice that the triangle inequality for D is trivial. To continue we need the following result.

Lemma A.0.3. *If ρ is a quasi-metric and $a, x_1, x_2, \ldots, x_n, b \in X$ then the following inequality holds*

$$\rho(a, b) \leq 2\rho(a, x_1) + 4\rho(x_1, x_2) + 4\rho(x_2, x_3) + \cdots + 4\rho(x_{n-1}, x_n) + 2\rho(x_n, b). \tag{A.2}$$

Proof. By contradiction assume the Lemma is false. Then there is some value of n for which (A.2) does not hold. Let N be the smallest such integer. Then

$$\rho(a, b) > 2\rho(a, x_1) + 4\rho(x_1, x_2) + 4\rho(x_2, x_3) + \cdots + 4\rho(x_{N-1}, x_N) + 2\rho(x_N, b). \tag{A.3}$$

while (A.2) holds for $n < N$. Notice that $N > 1$ because (A.2) holds for $n = 1$ by the generalized triangle property. The same property implies that for every x_r either

$$\rho(a, b) \leq 2\rho(a, x_r), \tag{A.4}$$

or

$$\rho(a, b) \leq 2\rho(x_r, b), \tag{A.5}$$

If $r = 1$, (A.4) does not hold because of (A.3), hence (A.5) does. Likewise (A.5) does not hold for $r = N$. Let R be the largest value of r for which (A.5) holds. Then $R < N$, and

$$\rho(a,b) \le 2\rho(x_R, b). \tag{A.6}$$

From the definition of R,

$$\rho(a,b) \le 2\rho(a, x_{R+1}). \tag{A.7}$$

Since (A.2) holds for $n < N$,

$$\rho(x_R, b) \le 2\rho(x_R, x_{R+1}) + 4\rho(x_{R+1}, x_{R+2}) + \cdots + 4\rho(x_{N-1}, x_N) + 2\rho(x_N, b), \tag{A.8}$$

and

$$\rho(a, x_{R+1}) \le 2\rho(a, x_1) + 4\rho(x_1, x_2) + \cdots + 4\rho(x_{R-1}, x_R) + 2\rho(x_R, x_{R+1}), \tag{A.9}$$

Adding (A.8) and (A.9) and combining with (A.6) and (A.7) gives

$$\rho(a,b) \le 2\rho(a, x_1) + 4\rho(x_1, x_2) + 4\rho(x_2, x_3) + \cdots + 4\rho(x_{N-1}, x_N) + 2\rho(x_N, b)$$

which contradicts (A.3). □

Proposition A.0.4 (Frink [35]). *The function D is a metric on X such that*

$$D(x,y) \le \rho(x,y) \le 4D(x,y). \tag{A.10}$$

Moreover, the topology of D is the one induced by ρ.

Proof. The first inequality (A.10) follows by definition. The second one is a consequence of (A.2). To prove that D is a metric use (A.10). The triangle inequality for D is trivial. □

Bibliography

[1] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds with negative curvature*, Proc. Steklov Inst. Mathematics **90** (1967).

[2] N. Aoki, *The set of axiom A diffeomorphisms with no cycles*, Bulletin Brazilian Mathematical Society **23** (1992), 21–65.

[3] N. Aoki and K. Hiraide, *Topological theory of dynamical systems*, North-Holland, 1994.

[4] A. Artigue, J. Brum, and R. Potrie, *Local product structure for expansive homeomorphisms*, Topology Appl. **156** (2009), 674–685.

[5] A. Artigue, *Expansive flows of surfaces*, Disc. & cont. dyn. sys. **33** (2013), no. 2, 505–525.

[6] ______, *Hyper-expansive homeomorphisms*, Publicaciones Matemáticas del Uruguay **14** (2013), 72–77.

[7] ______, *Positive expansive flows*, Topology and its Applications **165** (2014), 121–132.

[8] ______, *Kinematic expansive flows*, Ergodic Theory and Dynamical Systems **FirstView** (2014), 1–32, DOI 10.1017/etds.2014.65.

[9] A. Artigue, M. J. Pacífico, and J. L. Vieitez, *N-expansive homeomorphisms on surfaces*, Communications in Contemporary Mathematics (2014).

[10] A. Artigue, *Finite sets with fake observable cardinality*, Bull. Korean Math. Soc. **52** (2015), 323–333.

[11] A. Avila and A. Kocsard, *Cohomological Equations and Invariant Distributions for Minimal Circle Diffeomorphisms*, Duke Mathematical Journal **158** (2011), no. 3, 501–536.

[12] G. L. Baker and J. A. Blackburn, *The Pendulum: a case study in physics*, Oxford University Press, 2005.

[13] W. Bauer and K. Sigmund, *Topological dynamics of transformations induced on the space of probability measures*, Monatsh Math **79** (1975), 81–92.

[14] N. P. Bhatia and G. P. Szegö, *Dynamical Systems: Stability Theory and Applications*, Lect. Not. in Math., vol. 35, Springer-Verlag, 1967.

[15] G. D. Birkhoff, *Sur le problème restreint des trois corps*, Annali della R. Scuola Normale Superiore di Pisa **5** (1936), 1–42.

[16] R. Bowen, *Markov partitions and minimal sets for Axiom A diffeomorphisms*, Amer. J. Math **92** (1970), 903–918.

[17] R. Bowen and P. Walters, *Expansive one-parameter flows*, J. Diff. Eq. **12** (1972), 180–193.

[18] R. Bowen, *Entropy-expansive maps*, Trans. of the AMS **164** (1972), 323–331.

[19] ———, *Symbolic dynamics for hyperbolic flows*, Am. J. of Math. **95** (1973), 429–460.

[20] M. Brunella, *Expansive flows on Seifert manifolds and torus bundles*, Bol. Soc. Bras. Mat. **24** (1993), 89–104.

[21] B. F. Bryant, *Unstable self-homeomorphisms of a compact space*, Vanderbilt University Thesis, 1954.

[22] ———, *Expansive Self-Homeomorphisms of a Compact Metric Space*, Amer. Math. Monthly **69** (1962), 386–391.

[23] G. Cantor, *Uber unendliche line are Punktmannigfaltigkeiten, 5. Fortsetzung*, Vol. 21, 1883.

[24] M. Cerminara and J. Lewowicz, *Some open problems concerning expansive systems*, Rend. Istit. Mat. Univ. Trieste **42** (2010), 129–141.

[25] J. J. Charatonik, *History of continuum theory*, Kluwer Academic Publishers, 1998.

[26] C. Conley, *Isolated invariant sets and the Morse index*, AMS, 1978.

[27] E. M. Coven and M. Keane, *Every compact metric space that supports a positively expansive homeomorphism is finite*, IMS Lecture Notes Monogr. Ser., Dynamics & Stochastics **48** (2006), 304–305.

[28] T. Das, K. Lee, and M. Lee, *C^1-persistently continuum-wise expansive homoclinic classes and recurrent sets*, Topology and its Applications **160** (2013), 350–359.

[29] A. DeStefano and G. R. Hall, *An Example of a Universally Observable Flow on the Torus*, Siam J. Control Optim. **36** (1998), no. 4, 1207–1224.

[30] A. Fathi, *Expansiveness, hyperbolicity and Hausdorff dimension*, Commun. Math. Phys. **126** (1989), 249–262.

[31] J. Franks, *Necessary conditions for stability of diffeomorphisms*, Trans. Amer. Math. Soc. **158** (1971), 301–308.

[32] J. Franks and C. Robinson, *A quasi-Anosov diffeomorphism that is not Anosov*, Trans. of the AMS **223** (1976), 267–278.

[33] M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rend. Circ. Mat. Palermo **22** (1906), 1–71.

[34] D. Fried, *Finitely presented dynamical systems*, Ergod. Th. Dynam. Sys. **7** (1987), 489–507.

[35] A. H. Frink, *Distance functions and the metrization problem*, Bull. Am. Math. Soc. **43** (1937), 133–142.

[36] S. Gan and L. Wen, *Nonsingular star flows satisfy Axiom A and the no-cycle condition*, Invent. math. **164** (2006), 279–315.

[37] M. Garcia and G. A. Hedlund, *The structure of minimal sets*, Bull. Amer. Math. Soc. **54** (1948), 954–964.

[38] W. H. Gottschalk, *Almost periodicity, equi-continuity and total boundedness*, Bull. Amer. Math. Soc. **52** (1946), 633–636.

[39] A. A. Gura, *Separating diffeomorphisms of the torus*, Mat. Zametki **18** (1975), 41–49.

[40] ———, *Horocycle flow on a surface of negative curvature is separating*, Mat. Zametki **36** (1984), 279–284.

[41] C. Gutierrez, *Smoothing continuous flows on two-manifolds and recurrences*, Ergod. Th & Dynam. Sys. **6** (1986), 17–14.

[42] H. Hahn, *Mengentheoretische Charakterisierung der stetigen Kurve*, Sitzungsberichte Akademie der Wissenschaften in Wien, Mathematischnaturwissenschaftliche Klasse, Abteilung IIa **123** (1914), 2433–2489.

[43] ———, *Uber die allgemeine ebene Punktmenge, die stetiges Bild einer Streckeist*, lahresbericht der Deutschen Mathematiker- Vereinigung **23** (1914)), 318–322.

[44] P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons Inc., New York, 1964.

[45] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914.

[46] S. Hayashi, *Diffeomorphisms in $F^1(M)$ satisfy Axiom A*, Ergodic Theory Dyn. Syst. **12** (1992), 233–253.

[47] L. F. He and G. Z. Shan, *The nonexistence of expansive flow on a compact 2-manifold*, Chinese Ann. Math. Ser. B **12** (1991), 213–218.

[48] G. A. Hedlund and M. Morse, *Symbolic dynamics*, Amer. J. Math. **60** (1938), 815–866.

[49] G. A. Hedlund, *Sturmian minimal sets*, Amer. J. Math. **66** (1944), 605–620.

[50] K. Hiraide, *Expansive homeomorphisms with the pseudo-orbit tracing property on compact surfaces*, J. Math. Soc. Japan **40** (1988), 123–137.

[51] ———, *Expansive homeomorphisms with the pseudo-orbit tracing property of n-tori*, J. Math. Soc. Japan **41** (1989), 357–389.

[52] ———, *Expansive homeomorphisms of compact surfaces are pseudo-Anosov*, Osaka J. Math. **27** (1990), 117–162.

[53] J. Hocking and G. Young, *Topología*, Editorial Reverté S. A., 1966.

[54] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Univ. Press, 1948.

[55] M. Hurley, *Lyapunov Functions and Attractors in Arbitrary Metric Spaces*, Proc. of the Am. Math. Soc. **126** (1998), 245–256.

[56] T. Inaba and S. Matsumoto, *Nonsingular expansive flows on 3-manifolds and foliations with circle prong singularities*, Japan. J. Math. **16** (1990), 329–340.

[57] J. F. Jakobsen y W. R. Utz, *The non-existence of expansive homeomorphisms on a closed 2-cell*, Pacific J. Math. **10** (1960), no. 4, 1319–1321.

[58] Z. Janiszewski, *O rozcinaniu plaszczyzny przez continua*, Prace Matematyczno-Fizyczne **26** (1913), 11–63.

[59] H. Kato, *The nonexistence of expansive homeomorphisms of Peano continua in the plane*, Topology Appl. **34** (1990), 161–165.

[60] ______, *Expansive homeomorphisms in continuum theory*, Topology and its applications **45** (1992), 223–243.

[61] ______, *Continuum-wise expansive homeomorphisms*, Canad. J. Math. **45** (1993), no. 3, 576–598.

[62] ______, *Concerning continuum-wise fully expansive homeomorphisms of continua*, Topology and its Applications **53** (1993), 239–258.

[63] H. Kato and J. Park, *Expansive homeomorphisms of countable compacta*, Top. and its App. **95** (1999), 207–216.

[64] K. Kawamura, *A direct proof that each Peano continuum with a free arc admits no expansive homeomorphism*, Tsukuba J. Math. **12** (1988), 521–524.

[65] H. Keynes and J. Robertson, *Generators for topological entropy and expansiveness*, Mathematical systems theory **3** (1969), 51–59.

[66] H. B. Keynes and M. Sears, *$\mathcal{F}$-expansive transformation group*, General topology and its applications (1979), 67–85.

[67] H. Keynes and M. Sears, *Real-expansive flows and topological dimension*, Ergodic Theory and Dynamical Systems **1** (1981), 179–195.

[68] A. Kocsard, *Toward the classification of cohomology-free vector fields*, Thesis Impa, 2007.

[69] M. Komuro, *Expansive properties of Lorenz attractors*, The Theory of dynamical systems and its applications to nonlinear problems (1984), 4–26.

[70] K. Kuratowski, *Topology*, Vol. II, Academic Press, New York and London, 1968.

[71] P. Lam, *On expansive transformation groups*, Trans. AMS **150** (1970), 131–138.

[72] J. Lewowicz, *Lyapunov Functions and Topological Stability*, J. Diff. Eq. **38** (1980), 192–209.

[73] ______, *Lyapunov functions and stability of geodesic flows*, Geometric Dynamics (Rio de Janeiro 1981), 1981, pp. 463–479.

[74] ______, *Expansive homeomorphisms of surfaces*, Bol. Soc. Bras. Mat. **20** (1989), 113–133.

[75] ______, *Dinámica de los homeomorfismos expansivos*, Monografías del IMCA, 2003.

[76] A. M. Lyapunov, *The General Problem of the Stability of Motion*, Doctoral dissertation, Univ. Kharkov, 1892.

[77] R. Mañé, *Expansive diffeomorphisms*, Dynamical Systems—Warwick 1974, 1975, pp. 162-174.

[78] ______, *Expansive homeomorphisms and topological dimension*, Trans. of the AMS **252** (1979), 313–319.

[79] J. L. Massera, *On Liapunoff's Conditions of Stability*, Ann. of Math. **50** (1949), no. 3, 705–721.

[80] ______, *The meaning of stability*, Bol. Fac. Ingen. Agrimens. Montevideo **8** (1964), 405–429.

[81] S. Matsumoto, *Kinematic expansive suspensions of irrational rotations on the circle*, arXiv:1412.0399 (2014).

[82] S. Mazurkiewicz, *On the arithmetization of continua*, C. R. Soc. Sc. de Varsovie **6** (1913), 305–311; part II, 941–945.

[83] ______, *Sur les lignes de Jordan*, Fund. Math. **1** (1920), 166–209.

[84] ———, *Sur le type de dimension de l'hyperespace d'un continu*, C. R. Soc. Sc. Varsovie **24** (1931), 191–192.

[85] C. A. Morales, *Measure expansive systems*, Preprint IMPA (2011).

[86] C. A. Morales and V. F. Sirvent, *Expansive measures*, 29o Colóq. Bras. Mat., IMPA, 2013.

[87] C. A. Morales, *A generalization of expansivity*, Discrete Contin. Dyn. Syst. **32** (2012), no. 1, 293–301.

[88] K. Moriyasu, K. Sakai, and W. Sun, *C^1-stably expansive flows*, Journal of Differential Equations **213** (2005), 352–367.

[89] C. Mouron, *Tree-like continua do not admit expansive homeomorphisms*, Proc. Amer. Math. Soc. **130** (2002), 3409–3413.

[90] ———, *An expansive homeomorphism on a two-dimensional planar continuum*, Topology Proc. **27** (2003), 559–573.

[91] ———, *Expansive homeomorphisms and plane separating continua*, Topology and its applications **155** (2008), 1000–1012.

[92] S. Nadler Jr., *Hyperspaces of Sets*, Marcel Dekker Inc. New York and Basel, 1978.

[93] A. Nagar and P. Sharma, *Topological dynamics on hyperspaces*, Applied general topology **11** (2010), 1–19.

[94] T. O'Brien and W. L. Reddy, *Each compact orientable surface of positive genus admits an expansive homeomorphism*, Pacific J. Math. **35** (1970), 533–806.

[95] M. J. Pacifico, E. R. Pujals, M. Sambarino, and J. L.Vieitez, *Robustly expansive codimension-one homoclinic classes are hyperbolic*, Ergodic Theory Dynam. Systems **29** (2009), 179–200.

[96] M. J. Pacifico, E. R. Pujals, and J. L. Vieitez, *Robustly expansive homoclinic classes*, Ergodic Theory Dynam. Systems **25** (2005), 271–300.

[97] M. J. Pacifico and J. L. Vieitez, *Entropy expansiveness and domination for surface diffeomorphisms*, Rev. Mat. Complut. **21** (2008), no. 2, 293–317.

[98] ———, *Robust entropy expansiveness implies generic domination*, Nonlinearity **23** (2010), 1971–1990.

[99] J. Palis and F. Takens, *Hyperbolicity and Sensitive-Chaotic Dynamics at Homoclinic Bifurcations*, Cambridge University Press, 1993.

[100] M. Paternain, *Expansive flows and the fundamental group*, Bull. Braz. Math. Soc. **24** (1993), no. 2, 179–199.

[101] ———, *Expansive geodesic flows on surfaces*, Ergodic Theory Dynam. Systems **13** (1993), 153–165.

[102] ———, *Expansivity and length expansivity for geodesic flows on surfaces*, Pitman Res. Notes Math. Ser. **285** (1993), 195–210.

[103] M. M. Peixoto, *Structural stability on two-dimensional manifolds*, Topology **1** (1962), 101–120.

[104] A. A. Petrov and S. Y. Pilyugin, *Lyapunov functions, shadowing, and topological stability*, Arxiv (2013).

[105] R. V. Plykin, *On the geometry of hyperbolic attractors of smooth cascades*, Russian Math. Survey **39** (1984), 85–131.

[106] W. L. Reddy, *The existence of expansive homeomorphisms on manifolds*, Duke Math. J. **32** (1965), 565–765.

[107] ______, *Pointwise expansion homeomorphisms*, J. Lond. Math. Soc. **2** (1970), 232–236.

[108] ______, *On positively expansive maps*, Mathematical systems theory **6** (1972), 76–81.

[109] ______, *Expansive canonical coordinates are hyperbolic*, Topology and its applications **15** (1983), 205–210.

[110] C. Robinson, *A quasi-Anosov flow that is not Anosov*, Indiana University Mathematics Journal **25** (1976), 763–767.

[111] ______, *Dynamical Systems*, CRC Press, 1995.

[112] J. Rodriguez Hertz, *There are no stable points for continuum-wise expansive homeomorphisms*, Pre. Mat. Urug. 65 (2002).

[113] M. E. Rudin, *A Topological Characterization of Sets of Real Numbers*, Pacific Journal of Mathematics **7** (1957), no. 2, 1185–1186.

[114] R. O. Ruggiero, *Persistently expansive geodesic fows*, Comm. Math. Physics **140** (1991), 203–215.

[115] ______, *Expansive geodesic flows: from the work of J. Lewowicz in low dimensions*, Publ. Mat. Urug. (2013), 25–59.

[116] K. Sakai, *Hyperbolic metrics of expansive homeomorphisms*, Topology and its applications **63** (1995), 263–266.

[117] K. Sakai, *Continuum-wise expansive diffeomorphisms*, Publicacions Matemàtiques **41** (1997), 375–382.

[118] S. Schwartzman, *On transformation groups*, Dissertation, Yale University (1952).

[119] M. Sears, *Expansiveness on locally compact spaces*, Mathematical systems theory **7** (1973), 377–382.

[120] M. Shub, *Global stability of dynamical systems*, Springer-Verlag, 1987.

[121] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747-817.

[122] ______, *The Ω-Stability Theorem*, Proc. Symp. Pure Math. **14** (1970), 289–297.

[123] R. F. Thomas, *Stability properties of one parameter flows*, Proc. London Math. Soc. **3** (1982), 479–505.

[124] ______, *Topological stability: some fundamental properties*, J. Diff. Eq. **59** (1985), 103–122.

[125] ______, *Entropy of expansive flows*, Erg. Th. & Dyn. Sys **7** (1987), 611–625.

[126] ______, *Topological entropy of fixed-point free flows*, Trans. Amer. Math. Soc. **319** (1990), 601–618.

[127] R. Ures, *On expansive covering maps*, Publicaciones Matemáticas del Uruguay **3** (1990), 59–67.

[128] W. R. Utz, *Unstable homeomorphisms*, Proc. Amer. Math. Soc. **1** (1950), no. 6, 769–774.

[129] M. Viana, *Ergodic theory of interval exchange maps*, Rev. Mat. Complut. **19** (2006), 7–100.

[130] J. L. Vieitez, *Three-dimensional expansive homeomorphisms*, Pitman Res. Notes Math. Ser. **285** (1993), 299–323.

[131] ———, *Three-dimensional expansive diffeomorphisms with homoclinic points*, Bol. Soc. Brasil. Mat. **27** (1996), 55–90.

[132] ———, *Expansive homeomorphisms and hyperbolic diffeomorphisms on 3-manifolds*, Ergodic Theory Dynam. Systems **16** (1996), 591–622.

[133] ———, *Lyapunov functions and expansive diffeomorphisms on 3D-manifolds*, Ergod Theor Dyn Syst **22** (2002), 601-632.

[134] P. Walters, *On the pseudo orbit tracing property and its relationship to stability.*

[135] H. Whitney, *Regular families of curves*, Ann. of Math. **34** (1933), 244–270.

[136] R. L. Wilder, *Topology of Manifolds*, Vol. 32, AMS Colloquium Publications, 1979.

[137] R. F. Williams, *A note on unstable homeomorphisms*, Proc. Amer. Math. Soc. **6** (1955), 308–309.

Index